Temporal Logics for the Specification of Hyperproperties

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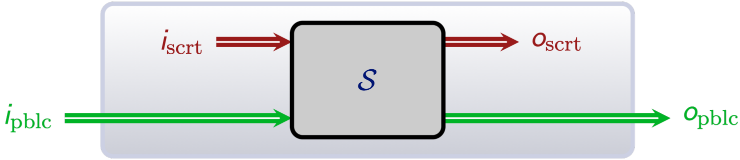
NetVAS 22

Motivation



i_{scrt}
o_{scrt}
$\mathcal{S}$
i_{pblc}
o_{pblc}

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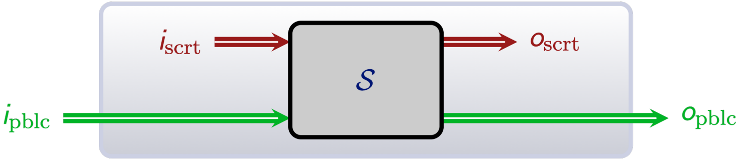

Trace-based view on $\mathcal{S}$: observe execution traces, i.e., infinite sequences over 2^{AP} for some set AP of atomic propositions.

$$\{\mathtt{init}, \mathtt{i}_{\mathsf{pblc}}\} \quad \{\mathtt{i}_{\mathsf{scrt}}\} \quad \{\mathtt{i}_{\mathsf{pblc}}\} \quad \{\mathtt{i}_{\mathsf{scrt}}, \mathtt{o}_{\mathsf{pblc}}, \mathtt{term}\} \quad \cdots$$

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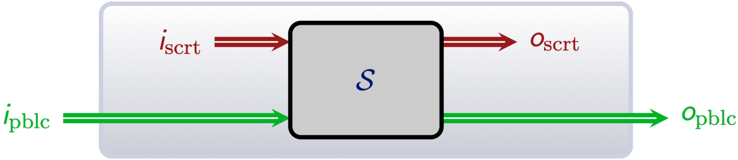

Typical requirements:

- $\mathcal{S}$ terminates

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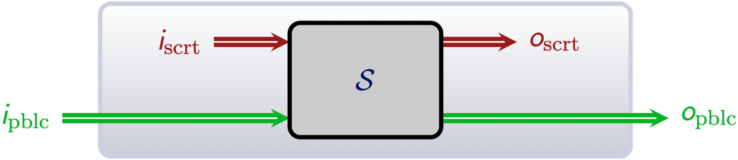

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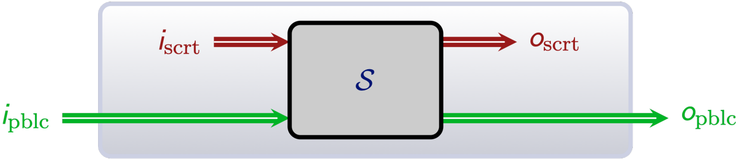

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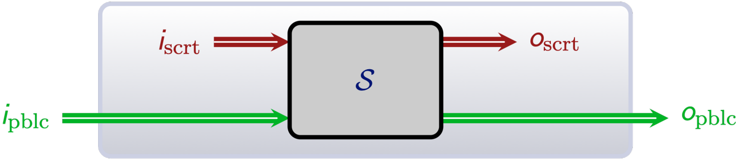

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- Noninterference: for all traces t, t' of $\mathcal{S}$

$$t =_{i_{\mathrm{pblc}}} t' \quad \text{implies} \quad t =_{o_{\mathrm{pblc}}} t'$$

Trace Properties vs. Hyperproperties

Definition

A trace property $T \subseteq (2^{\mathrm{AP}})^\omega$ is a set of traces. A system $\mathcal{S}$ satisfies T, if $\mathrm{Traces}(\mathcal{S}) \subseteq T$.

Example: The set of traces where `term` holds at least once.

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Definition

A hyperproperty $H \subseteq 2^{(2^{\mathrm{AP}})^\omega}$ is a set of sets of traces. A system $\mathcal{S}$ satisfies H if $\mathrm{Traces}(\mathcal{S}) \in H$.

Example: The set $\{T \subseteq T_n \mid n \in \mathbb{N}\}$ where T_n is the trace property containing the traces where `term` holds at least once within the first n positions.

LTL in One Slide

Syntax

$$\varphi ::= a \mid \neg\varphi \mid \varphi \vee \varphi \mid \mathbf{X}\,\varphi \mid \varphi\,\mathbf{U}\,\varphi \qquad \text{where } a \in \mathrm{AP}$$

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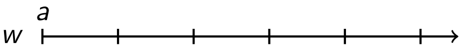
- $w \models a$:



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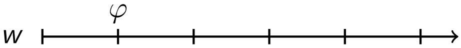

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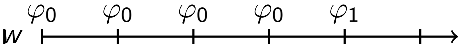

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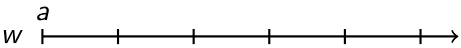

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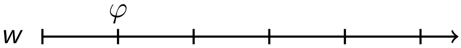

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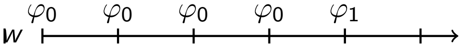

Syntactic Sugar

- $\mathbf{F}\,\psi = \mathbf{true}\,\mathbf{U}\,\psi$
- $\mathbf{G}\,\psi = \neg\mathbf{F}\,\neg\psi$

The Virtues of LTL

LTL is the most important specification language for reactive systems and has many desirable properties:

1. Every satisfiable LTL formula is satisfied by an ultimately periodic trace, i.e., by a finitely-represented model.
2. LTL and FO[$<$] are expressively equivalent.
3. LTL satisfiability and model-checking are PSpace-complete.

HyperLTL

HyperLTL = LTL + trace quantification

$$\varphi ::= \exists \pi.\ \varphi \mid \forall \pi.\ \varphi \mid \psi$$
$$\psi ::= a_\pi \mid \neg\psi \mid \psi \vee \psi \mid \mathbf{X}\,\psi \mid \psi\,\mathbf{U}\,\psi$$

where $a \in \mathrm{AP}$ and $\pi \in \mathcal{V}$ (trace variables).

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- Prenex normal form, but
- closed under boolean combinations.

Examples

- $\mathcal{S}$ is input-deterministic: for all traces t, t' of $\mathcal{S}$

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- $\mathcal{S}$ terminates within a uniform time bound.
Not expressible in HyperLTL.

Applications

- Uniform framework for information-flow control
  - Does a system leak information?
- Symmetries in distributed systems
  - Are clients treated symmetrically?
- Error resistant codes
  - Do codes for distinct inputs have at least Hamming distance d?
- Software doping
  - Think emission scandal in automotive industry
- Network verification?

There are prototype tools for model checking, satisfiability checking, runtime verification, and synthesis.

Another Example

Fix $\mathrm{AP} = \{a\}$ and consider the conjunction φ of

- $\forall \pi.\ (\neg a_\pi)\,\mathbf{U}\,(a_\pi \wedge \mathbf{X}\,\mathbf{G}\,\neg a_\pi)$

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$$\{a\} \quad \emptyset \quad \emptyset \quad \emptyset \quad \emptyset \quad \emptyset \quad \emptyset \quad \emptyset \quad \cdots$$

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The unique model of φ is $\{\emptyset^n\,\{a\}\,\emptyset^\omega \mid n \in \mathbb{N}\}$.

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Consequence:

There is a satisfiable HyperLTL sentence that is not satisfied by any finite set of traces.

Undecidability

The HyperLTL satisfiability problem:

Given φ, is there a non-empty set T of traces with $T \models \varphi$?

Theorem (Fortin et. al '21)

HyperLTL satisfiability is Σ^1_1-complete (i.e., highly undecidable).

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Fine-grained analysis:

Theorem (Finkbeiner & Hahn '16)

1. $\forall\exists$-*HyperLTL satisfiability is undecidable.*
2. $\exists^*$-*HyperLTL satisfiability is* PSpace-*complete.*
3. $\forall^*$-*HyperLTL satisfiability is* PSpace-*complete.*
4. $\exists^*\forall^*$-*HyperLTL satisfiability is* ExpSpace-*complete.*

Model-Checking

The HyperLTL model-checking problem:

Given a transition system $\mathcal{S}$ and φ, does $\mathrm{Traces}(\mathcal{S}) \models \varphi$?

Theorem (Clarkson et al. '14)

The HyperLTL model-checking problem is decidable.

Corollary (Mascle & Z. '20)

The HyperLTL model-checking problem is TOWER-hard, even for a fixed transition system with 5 states and formulas without nested operators.

Model-Checking

Proof:

- Consider $\varphi = \exists\pi_1.\ \forall\pi_2.\ \ldots \exists\pi_{k-1}.\ \forall\pi_k.\ \psi$.
- Rewrite as $\exists\pi_1.\ \neg\exists\pi_2.\ \neg \ldots \exists\pi_{k-1}.\ \neg\exists\pi_k.\ \neg\psi$.

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- By induction over quantifier prefix construct non-determinstic Büchi automaton $\mathcal{A}$ with $L(\mathcal{A}) \neq \emptyset$ iff $\mathrm{Traces}(\mathcal{S}) \models \varphi$.
  - Induction start: build automaton for LTL formula obtained from $\neg\psi$ by replacing a_{π_j} by a_j.
  - For $\exists\pi_j\theta$ restrict automaton for θ in dimension j to traces of $\mathcal{S}$.
  - For $\neg\theta$ complement automaton for θ.

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$\Rightarrow$ Non-elementary complexity, but alternation-free fragments are as hard as LTL.

Conclusion

HyperLTL behaves quite differently than LTL:

- The models of HyperLTL are rather not well-behaved, i.e., in general (countably) infinite, non-regular, and non-periodic.
- Satisfiability is in general undecidable.
- Model-checking is decidable, but non-elementary.

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- Satisfiability is in general undecidable.
- Model-checking is decidable, but non-elementary.

But with the feasible problems, you can do exciting things:
HyperLTL is a powerful tool for information security and beyond:

- Information-flow control
- Symmetries in distributed systems
- Error resistant codes
- Software doping
- Soon: Network verification