Martin Zimmermann
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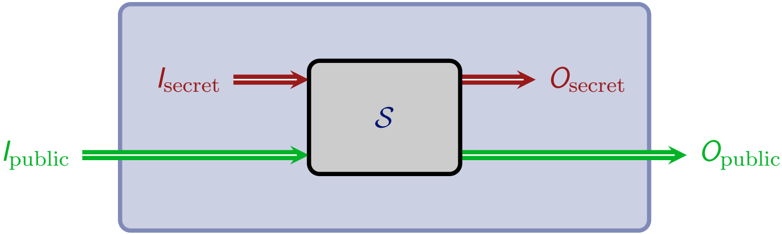
Logics for Hyperproperties

UniVr/UniUd Summer School on Formal Methods for Cyber-Physical Systems, Udine, August 30, 2023

Hyperproperties



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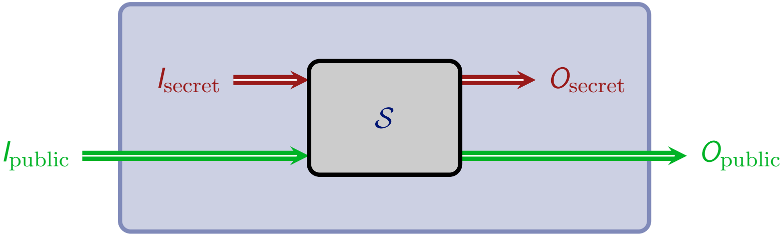

- The system $\mathcal{S}$ is input-deterministic: for all traces t, t' of $\mathcal{S}$

$$t =_I t' \quad \text{implies} \quad t =_O t'$$

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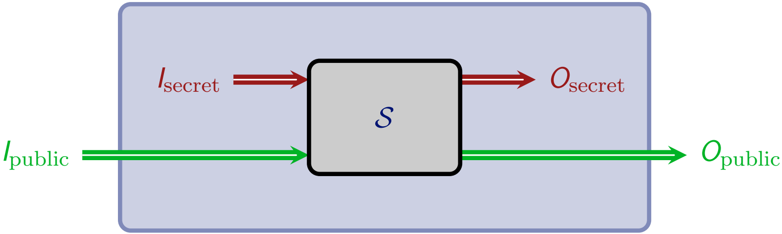

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$$t =_I t' \quad \text{implies} \quad t =_O t'$$

- Noninterference: for all traces t, t' of $\mathcal{S}$

$$t =_{I_{\text{public}}} t' \quad \text{implies} \quad t =_{O_{\text{public}}} t'$$

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- A system $\mathcal{S}$ satisfies a hyperproperty H, if $\mathrm{Traces}(\mathcal{S}) \in H$.

Example: Noninterference as hyperproperty:

$$\{T \subseteq \mathrm{Traces}(\mathrm{AP}) \mid \forall t, t' \in T : t =_{I_{\mathrm{public}}} t' \Rightarrow t =_{O_{\mathrm{public}}} t'\}$$

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Specification languages for hyperproperties

HyperLTL: Extend LTL by trace quantifiers.

HyperCTL*: Extend CTL* by trace quantifiers.

Outline

1. HyperLTL
2. The Models Of HyperLTL
3. HyperLTL Satisfiability
4. HyperLTL Model-checking
5. The First-order Logic of Hyperproperties
6. Conclusion

Agenda

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LTL in One Slide

Syntax

$$\varphi ::= a \mid \neg\varphi \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \mathbf{X}\,\varphi \mid \varphi\,\mathbf{U}\,\varphi$$

where $a \in \mathrm{AP}$ (atomic propositions).

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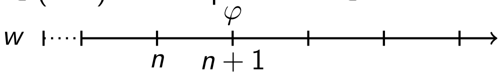
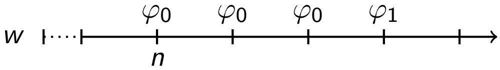
- $w, n \models \mathbf{X}\,\varphi$: 

- $w, n \models \varphi_0\,\mathbf{U}\,\varphi_1$: 

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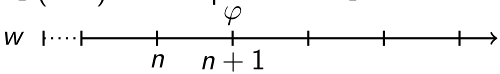

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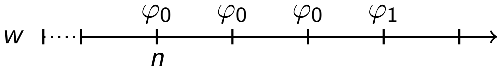

Syntactic Sugar

- $\mathbf{F}\,\psi = \mathbf{true}\,\mathbf{U}\,\psi$
- $\mathbf{G}\,\psi = \neg\mathbf{F}\,\neg\psi$

HyperLTL

HyperLTL = LTL + trace quantification

$$\varphi ::= \exists \pi.\ \varphi \mid \forall \pi.\ \varphi \mid \psi$$
$$\psi ::= a_{\pi} \mid \neg \psi \mid \psi \vee \psi \mid \psi \wedge \psi \mid \mathbf{X}\, \psi \mid \psi\, \mathbf{U}\, \psi$$

where $a \in \mathrm{AP}$ (atomic propositions) and $\pi \in \mathcal{V}$ (trace variables).

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- Prenex normal form, but
- closed under boolean combinations.

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- Trace assignment: partial mapping $\Pi\colon \mathcal{V} \to (2^{\mathrm{AP}})^\omega$
- Empty trace assignment: $\Pi_\emptyset$

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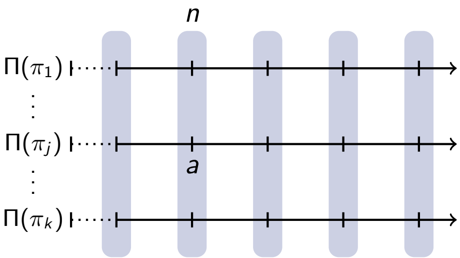

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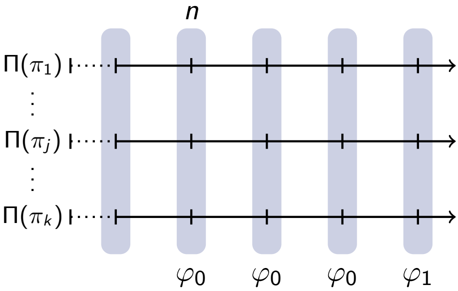

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- $T, \Pi, n \models \exists \pi.\, \varphi$ if $T, \Pi[\pi \mapsto t], n \models \varphi$ for some $t \in T$.
- $T, \Pi, n \models \forall \pi.\, \varphi$ if $T, \Pi[\pi \mapsto t], n \models \varphi$ for all $t \in T$.

T is a model of a sentence φ, written $T \models \varphi$, if $T, \Pi_\emptyset, 0 \models \varphi$.

Semantics: Example

$$\varphi = \forall\pi.\, \forall\pi'.\, \mathbf{G}\, \mathtt{on}_\pi \leftrightarrow \mathtt{on}_{\pi'}$$

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& \mathtt{on} \in t(n) \Leftrightarrow \mathtt{on} \in t'(n)
\end{aligned}$$

Applications

- Uniform framework for information-flow control
  - Does a system leak information?
- Symmetries in distributed systems
  - Are clients treated symmetrically?
- Error resistant codes
  - Do codes for distinct inputs have at least Hamming distance d?
- Software doping
  - Think emission scandal in automotive industry

The Virtues of LTL

LTL has many desirables properties:

1. Every satisfiable LTL formula is satisfied by an ultimately periodic trace, i.e., by a finite and finitely-represented model.
2. LTL satisfiability and model-checking are PSPACE-complete.
3. LTL and FO[$<$] are expressively equivalent.

Which properties does HyperLTL retain ?

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Fix $\mathrm{AP} = \{a\}$ and consider the conjunction φ of

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Proof

- W.l.o.g. $\varphi = \forall \pi_0.\ \exists \pi'_0. \cdots \forall \pi_k.\ \exists \pi'_k.\ \psi$ with quantifier-free ψ.
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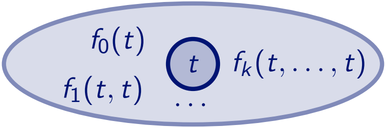

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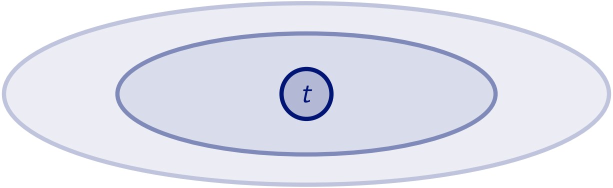

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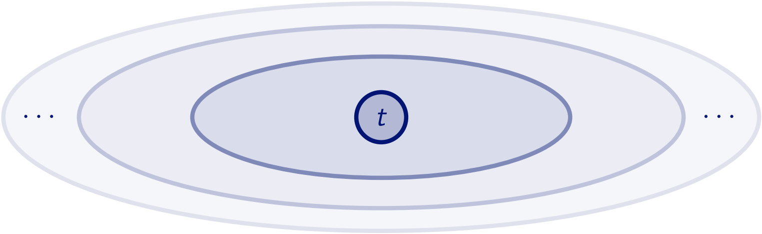

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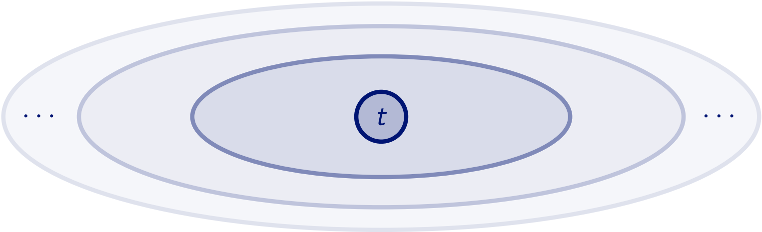

The limit is a model of φ and countable.

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Then, $T \cap \{a\}^*\{b\}^*\emptyset^\omega = \{\{a\}^n\{b\}^n\emptyset^\omega \mid n \in \mathbb{N}\}$ is not ω-regular.

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One can even encode the prime numbers in HyperLTL!

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Given φ, is there a non-empty set T of traces with $T \models \varphi$?

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By a reduction from Post's correspondence problem.

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A solution:

b	b	a	a	b	b	b	a	a
b	b	a	a	b	b	b	a	a

Undecidability

The HyperLTL satisfiability problem:

Given φ, is there a non-empty set T of traces with $T \models \varphi$?

Theorem

HyperLTL satisfiability is undecidable.

Proof:

By a reduction from Post's correspondence problem.

Example

Blocks (a, baa) (ab, aa) (bba, bb)

A solution:

b	b	a	a	b	b	b	a	a
b	b	a	a	b	b	b	a	a

Undecidability

1. There is a (solution) trace where top matches bottom.

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$$\begin{array}{cccccccccc} \{b\} & \{b\} & \{a\} & \{a\} & \{b\} & \{b\} & \{b\} & \{a\} & \{a\} & \emptyset^\omega \\ \{b\} & \{b\} & \{a\} & \{a\} & \{b\} & \{b\} & \{b\} & \{a\} & \{a\} & \emptyset^\omega \end{array}$$

Undecidability

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3. For every non-empty trace, the trace obtained by removing the first block also exists.

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Decidability

Theorem

$\exists^*$-*HyperLTL satisfiability is* $\text{PS}{\small\text{PACE}}$*-complete.*

Decidability

Theorem

$\exists^*$-*HyperLTL satisfiability is* $\mathrm{PS{\small PACE}}$-*complete.*

Proof:

- Membership:
  - Consider $\varphi = \exists\pi_0 \ldots \exists\pi_k.\, \psi$.
  - Obtain ψ' from ψ by replacing each a_{π_j} by a fresh proposition a_j.
  - Then: φ and the LTL formula ψ' are equi-satisfiable.
- Hardness: trivial reduction from LTL satisfiability

Decidability

Theorem

$\forall^*$*-HyperLTL satisfiability is* PSPACE*-complete.*

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$\forall^*$-*HyperLTL satisfiability is* $\textsc{PSpace}$-*complete.*

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$\exists^*\forall^*$-*HyperLTL satisfiability is* $\textsc{ExpSpace}$-*complete.*

Decidability

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$\exists^*\forall^*$-*HyperLTL satisfiability is* $\textsc{ExpSpace}$-*complete.*

Proof:

- Membership:
  - Consider $\varphi = \exists\pi_0 \ldots \exists\pi_k. \forall\pi'_0 \ldots \forall\pi'_\ell. \psi$.
  - Let

$$\varphi' = \exists\pi_0 \ldots \exists\pi_k \bigwedge_{j_0=0}^{k} \cdots \bigwedge_{j_\ell=0}^{k} \psi_{j_0,\ldots,j_\ell}$$

    where $\psi_{j_0,\ldots,j_\ell}$ is obtained from ψ by replacing each occurrence of π'_i by π_{j_i}.
  - Then: φ and φ' are equi-satisfiable.
- Hardness: encoding of exponential-space Turing machines.

Further Results

HyperLTL implication checking: given φ and φ', does, for every T, $T \models \varphi$ imply $T \models \varphi'$?

Lemma

φ does not imply φ' iff $(\varphi \wedge \neg\varphi')$ is satisfiable.

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Lemma

φ does not imply φ' iff $(\varphi \wedge \neg\varphi')$ is satisfiable.

Corollary

Implication checking for alternation-free HyperLTL formulas is $\textsc{ExpSpace}$*-complete.*

Tool **EAHyper**:

- satisfiability, implication, and equivalence checking for HyperLTL

Latest Results

Recently, the exact complexity of HyperLTL satisfiability was settled: it is highly undecidable.

Theorem (Fortin, Kuijer, Totzke, Z. 2021)

HyperLTL satisfiability is Σ_1^1-complete.

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HyperLTL satisfiability is Σ_1^1-complete.

Corollary (Fortin, Kuijer, Totzke, Z. 2021)

The membership problem for each level of the HyperLTL quantifier alternation hierarchy is Σ_1^1-complete.

Agenda

1. HyperLTL
2. The Models Of HyperLTL
3. HyperLTL Satisfiability
4. **HyperLTL Model-checking**
5. The First-order Logic of Hyperproperties
6. Conclusion

Model-Checking

The HyperLTL model-checking problem:

Given a transition system $\mathcal{S}$ and φ, does $\mathrm{Traces}(\mathcal{S}) \models \varphi$?

Theorem (Finkbeiner, Rabe, Sánchez 2015)

The HyperLTL model-checking problem is decidable.

Model-Checking

Proof:

- Consider $\varphi = \exists\pi_1.\,\forall\pi_2.\,\ldots\exists\pi_{k-1}.\,\forall\pi_k.\,\psi$.
- Rewrite as $\exists\pi_1.\,\neg\exists\pi_2.\,\neg\ldots\exists\pi_{k-1}.\,\neg\exists\pi_k.\,\neg\psi$.

Proof:

- Consider $\varphi = \exists \pi_1. \forall \pi_2. \ldots \exists \pi_{k-1}. \forall \pi_k. \psi$.
- Rewrite as $\exists \pi_1. \neg \exists \pi_2. \neg \ldots \exists \pi_{k-1}. \neg \exists \pi_k. \neg \psi$.
- By induction over quantifier prefix construct non-determinstic Büchi automaton $\mathcal{A}$ with $L(\mathcal{A}) \neq \emptyset$ iff $\mathrm{Traces}(\mathcal{S}) \models \varphi$.
  - Induction start: build automaton for LTL formula obtained from $\neg\psi$ by replacing a_{π_j} by a_j.
  - For $\exists \pi_j \theta$ restrict automaton for θ in dimension j to traces of $\mathcal{S}$.
  - For $\neg\theta$ complement automaton for θ.

Model-Checking

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    - For $\neg\theta$ complement automaton for θ.

$\Rightarrow$ Non-elementary complexity, but alternation-free fragments are as hard as LTL.

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1. HyperLTL
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First-order Logic vs. LTL

FO$[<]$: first-order order logic over signature $\{<\} \cup \{P_a \mid a \in \mathrm{AP}\}$ over structures with universe $\mathbb{N}$.

Theorem (Kamp '68, Gabbay et al. '80)

LTL and FO$[<]$ are expressively equivalent.

First-order Logic vs. LTL

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LTL and FO$[<]$ are expressively equivalent.

Example

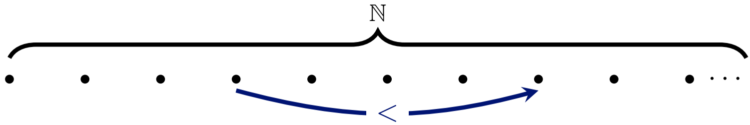
$$\forall x(P_q(x) \wedge \neg P_p(x)) \rightarrow \exists y(x < y \wedge P_p(y))$$

and

$$\mathbf{G}\,(q \rightarrow \mathbf{F}\,p)$$

are equivalent.

First-order Logic for Hyperproperties



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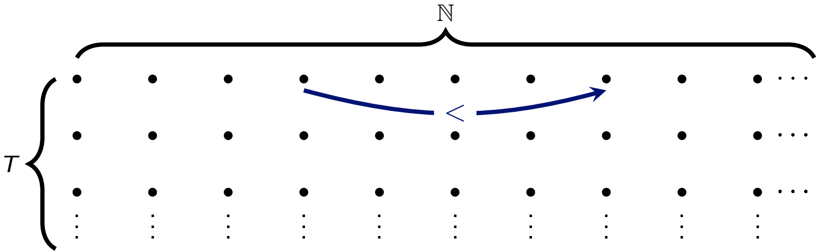

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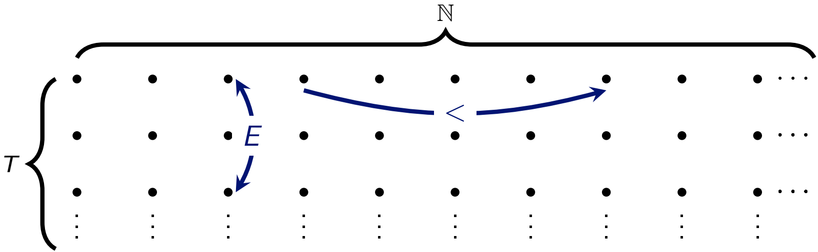

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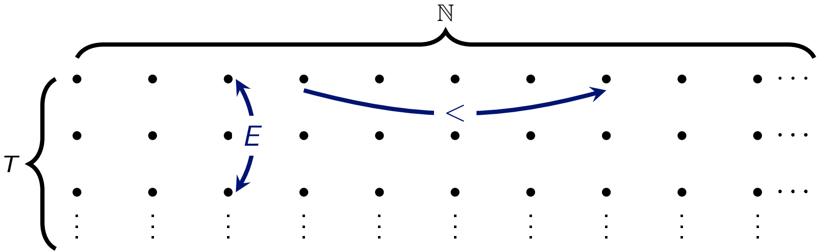
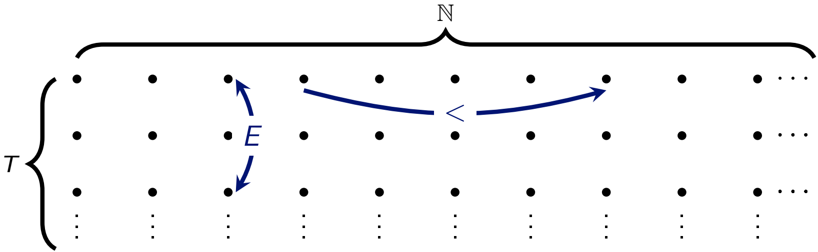


- FO[$<$, E]: first-order logic with equality over the signature $\{<, E\} \cup \{P_a \mid a \in \mathrm{AP}\}$ over structures with universe $T \times \mathbb{N}$.

Example

$$\forall x \forall x' \; E(x, x') \to (P_{\mathtt{on}}(x) \leftrightarrow P_{\mathtt{on}}(x'))$$

First-order Logic for Hyperproperties



- FO[$<$, E]: first-order logic with equality over the signature $\{<, E\} \cup \{P_a \mid a \in \mathrm{AP}\}$ over structures with universe $T \times \mathbb{N}$.

Proposition

For every HyperLTL sentence there is an equivalent FO[$<$, E] sentence.

A Setback

- Let φ be the following property of sets $T \subseteq (2^{\{p\}})^\omega$:

  There is an n such that $p \notin t(n)$ for every $t \in T$.

Theorem (Bozzelli et al. '15)

φ is not expressible in HyperLTL.

A Setback

- Let φ be the following property of sets $T \subseteq (2^{\{p\}})^\omega$:

  There is an n such that $p \notin t(n)$ for every $t \in T$.

Theorem (Bozzelli et al. '15)

φ is not expressible in HyperLTL.

- But, φ is easily expressible in FO$[<, E]$:

$$\exists x \, \forall y \; E(x, y) \rightarrow \neg P_p(y)$$

Corollary

$FO[<, E]$ strictly subsumes HyperLTL.

HyperFO

- $\exists^M x$ and $\forall^M x$: quantifiers restricted to initial positions.
- $\exists^G y \geq x$ and $\forall^G y \geq x$: if x is initial, then quantifiers restricted to positions on the same trace as x.

HyperFO

- $\exists^M x$ and $\forall^M x$: quantifiers restricted to initial positions.
- $\exists^G y \geq x$ and $\forall^G y \geq x$: if x is initial, then quantifiers restricted to positions on the same trace as x.

HyperFO: sentences of the form

$$\varphi = Q_1^M x_1. \cdots Q_k^M x_k.\ Q_1^G y_1 \geq x_{g_1}. \cdots Q_\ell^G y_\ell \geq x_{g_\ell}.\ \psi$$

- $Q \in \{\exists, \forall\}$,
- $\{x_1, \ldots, x_k\}$ and $\{y_1, \ldots, y_\ell\}$ are disjoint,
- every guard x_{g_j} is in$\{x_1, \ldots, x_k\}$, and
- ψ is quantifier-free over signature $\{<, E\} \cup \{P_a \mid a \in \mathrm{AP}\}$ with free variables in $\{y_1, \ldots, y_\ell\}$.

Equivalence

Theorem (Finkbeiner, Z. 2017)

HyperLTL and HyperFO are equally expressive.

Equivalence

Theorem (Finkbeiner, Z. 2017)

HyperLTL and HyperFO are equally expressive.

Proof

- From HyperLTL to HyperFO: structural induction.
- From HyperFO to HyperLTL: reduction to Kamp's theorem.

From HyperFO to HyperLTL

$$\forall x \forall x' \quad E(x, x') \rightarrow (P_{\mathtt{on}}(x) \leftrightarrow P_{\mathtt{on}}(x'))$$

From HyperFO to HyperLTL

$$\forall x \forall x' \quad E(x, x') \to (P_{\mathtt{on}}(x) \leftrightarrow P_{\mathtt{on}}(x'))$$

$$\forall^M x_1 \, \forall^M x_2 \quad \forall^G y_1 \geq x_1 \, \forall^G y_2 \geq x_2 E(y_1, y_2) \to (P_{\mathtt{on}}(y_1) \leftrightarrow P_{\mathtt{on}}(y_2))$$

From HyperFO to HyperLTL

$$\forall x \forall x' \quad E(x, x') \rightarrow (P_{\mathtt{on}}(x) \leftrightarrow P_{\mathtt{on}}(x'))$$

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$x_1 \mapsto$	$\{\mathtt{on}\}$	$\{\mathtt{on}\}$	$\emptyset$	$\{\mathtt{on}\}$	$\cdots$
$x_2 \mapsto$	$\{\mathtt{on}\}$	$\emptyset$	$\emptyset$	$\{\mathtt{on}\}$	$\cdots$

From HyperFO to HyperLTL

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$$\forall y_1 \, \forall y_2 \; (y_1 = y_2) \to (P_{(\texttt{on},1)}(y_1) \leftrightarrow P_{(\texttt{on},2)}(y_2))$$

$\{(\mathrm{on}, 1), (\mathrm{on}, 2)\}$	$\{(\mathrm{on}, 1)\}$	$\emptyset$	$\{(\mathrm{on}, 1), (\mathrm{on}, 2)\}$	$\ldots$

From HyperFO to HyperLTL

$$\forall x \forall x' \quad E(x, x') \rightarrow (P_{\mathtt{on}}(x) \leftrightarrow P_{\mathtt{on}}(x'))$$

$$\forall^G y_1 \geq x_1 \, \forall^G y_2 \geq x_2 E(y_1, y_2) \rightarrow (P_{\mathtt{on}}(y_1) \leftrightarrow P_{\mathtt{on}}(y_2))$$

$$\forall y_1 \, \forall y_2 \; (y_1 = y_2) \rightarrow (P_{(\mathtt{on},1)}(y_1) \leftrightarrow P_{(\mathtt{on},2)}(y_2))$$

$$\mathbf{G}\,((\mathrm{on}, 1) \leftrightarrow (\mathrm{on}, 2))$$

$\{(\mathrm{on}, 1),$ $(\mathrm{on}, 2)\}$	$\{(\mathrm{on}, 1)\}$	$\emptyset$	$\{(\mathrm{on}, 1),$ $(\mathrm{on}, 2)\}$	$\dots$

From HyperFO to HyperLTL

$$\forall x \forall x' \quad E(x, x') \rightarrow (P_{\mathtt{on}}(x) \leftrightarrow P_{\mathtt{on}}(x'))$$

$$\forall^M x_1 \, \forall^M x_2 \quad \forall^G y_1 \geq x_1 \, \forall^G y_2 \geq x_2 E(y_1, y_2) \rightarrow (P_{\mathtt{on}}(y_1) \leftrightarrow P_{\mathtt{on}}(y_2))$$

$$\forall y_1 \, \forall y_2 \; (y_1 = y_2) \rightarrow (P_{(\mathtt{on},1)}(y_1) \leftrightarrow P_{(\mathtt{on},2)}(y_2))$$

$$\mathbf{G}\,((\mathrm{on}, 1) \leftrightarrow (\mathrm{on}, 2))$$

$\{(\mathrm{on}, 1),$ $(\mathrm{on}, 2)\}$	$\{(\mathrm{on}, 1)\}$	$\emptyset$	$\{(\mathrm{on}, 1),$ $(\mathrm{on}, 2)\}$	$\ldots$

From HyperFO to HyperLTL

$$\forall x \forall x' \quad E(x, x') \rightarrow (P_{\mathtt{on}}(x) \leftrightarrow P_{\mathtt{on}}(x'))$$

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$$\forall y_1 \, \forall y_2 \; (y_1 = y_2) \rightarrow (P_{(\mathtt{on},1)}(y_1) \leftrightarrow P_{(\mathtt{on},2)}(y_2))$$

$$\mathbf{G}\,((\mathrm{on}, 1) \leftrightarrow (\mathrm{on}, 2))$$

$$\forall \pi_1 \, \forall \pi_2 \quad \mathbf{G}\,(\mathtt{on}_{\pi_1} \leftrightarrow \mathtt{on}_{\pi_2})$$

$\pi_1 \mapsto$ | $\{\mathtt{on}\}$ | $\{\mathtt{on}\}$ | $\emptyset$ | $\{\mathtt{on}\}$ | $\cdots$

$\pi_2 \mapsto$ | $\{\mathtt{on}\}$ | $\emptyset$ | $\emptyset$ | $\{\mathtt{on}\}$ | $\cdots$

Agenda

1. HyperLTL
2. The Models Of HyperLTL
3. HyperLTL Satisfiability
4. HyperLTL Model-checking
5. The First-order Logic of Hyperproperties
6. **Conclusion**

Conclusion

HyperLTL behaves quite differently than LTL:

- The models of HyperLTL are rather not well-behaved, i.e., in general (countably) infinite, non-regular, and non-periodic.
- Satisfiability is in general undecidable.
- Model-checking is decidable, but non-elementary.

Conclusion

HyperLTL behaves quite differently than LTL:

- The models of HyperLTL are rather not well-behaved, i.e., in general (countably) infinite, non-regular, and non-periodic.
- Satisfiability is in general undecidable.
- Model-checking is decidable, but non-elementary.

But with the feasible problems, you can do exciting things. HyperLTL is a powerful tool for information security and beyond:

- Information-flow control
- Symmetries in distributed systems
- Error resistant codes
- Software doping
- ...

References (1)

The basics

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First-order logic

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