Robust Probabilistic Temporal Logics

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Abstract

We robustify PCTL and PCTL*, the most important specification languages for probabilistic systems, and show that robustness does not increase the complexity of the model-checking problems.

1 Introduction

Specifications of reactive systems are typically implications $\varphi_a \rightarrow \varphi_g$ where φ_a is an environment assumption and φ_g is a system guarantee. Such a specification is satisfied whenever the assumption is violated, independently of the system's behaviour. Assume, for example, that both the assumption and the guarantee are invariants $\varphi_a = \Box\psi_a$ and $\varphi_g = \Box\psi_g$ for propositional formulas ψ_g and ψ_a. Then, the specification $\Box\psi_a \rightarrow \Box\psi_g$ is satisfied if the formula ψ_a is violated just once, even if the formula ψ_g never holds. Such a behaviour is clearly undesirable: the classical semantics of temporal logics are not robust to deal with violations of the environment assumption.

Considerable effort has been put into overcoming this "defect" to provide robust semantics for temporal logics. However, the notion of robustness is hard to formalize, which is witnessed by the plethora of incomparable notions of robustness in the literature on verification (see, e.g., the introduction of [1] for a recent overview). Here, we focus on an approach of Tabuada and Neider based on a novel, robust semantics for temporal logics, originally introduced for LTL [1]. The semantics has five truth-values, distinguishing satisfaction of a formula of the form $\Box\psi$ and four canonical ways it can be violated:

1. ψ does not hold always, but all but finitely often.
2. ψ does not hold all but finitely often, but infinitely often.
3. ψ does not hold infinitely often, but at least once.
4. ψ never holds.

Note that there is a natural order between these cases. Now, Tabuada and Neider defined the semantics of an implication $\Box\psi_a \rightarrow \Box\psi_g$ such that it is satisfied whenever the degree of violation of the guarantee $\Box\psi_g$ is not more severe than the violation of the assumption $\Box\psi_g$. Thus, the semantics indeed robustly handles violations of environment assumptions.

The resulting logic, called robust LTL (rLTL) has been extensively studied with very encouraging results: robustness can be added without increasing the complexity of model-checking and synthesis [1, 13, 10], robust semantics increases the usefulness of runtime monitoring [7], and rLTL can even be extended with increased expressiveness or timing constraints, again without an increase in complexity [11]. This approach towards robustness even extends to other temporal logics, e.g., branching-time logics like CTL and CTL* where robustness can again be added without increasing the complexity of the most important verification problems [8, 9].

Beyond the fact that this form of robustness comes for free, it only changes the semantics of the logics, but not the syntax. Furthermore, these logics are also evaluated over classical transition systems with the classical binary satisfaction relation for atomic propositions, i.e., robustness does not come from multi-valued semantics of the models (which might be hard to determine), but purely from the semantics. These aspects allow for a smooth transition from classical semantics to robust semantics for temporal logics.

However, these logics capture only robustness in the temporal dimension, i.e., they are concerned with a single execution. Statements like "99% of the executions satisfy answer each request eventually" require robustness in terms of the whole set of executions, which is orthogonal to the capabilities of the robust logics studied thus far.

To express such specifications, Hansson and Jonsson introduced probabilistic CTL (PCTL), which allows for probabilistic quantification [6]. For example, the property above is expressed by the formula $\mathcal{P}_{\geq .99}(\Box(q \rightarrow \Diamond p))$. Here, we show that PCTL (and even PCTL* [2], which is to PCTL what CTL* is to CTL), can be robustified for free, using the approach described above. In particular, we show that the complexity of model-checking does not increase, again showing the versatility of this approach to robustness.

2 Preliminaries

We denote the set of nonnegative integers by $\mathbb{N}$. Throughout the paper, we fix a finite set AP of atomic propositions we use to label our models and to build our formulas. For algorithmic purposes, we assume that all probabilities used in the following are rational.

2.1 Discrete-time Markov Chains

A discrete-time Markov chain (DTMC) $\mathcal{M} = (S, s_I, \delta, \ell)$ consists of a finite set S of states containing the initial state s_I, a (stochastic) transition function $\delta\colon S \times S \rightarrow [0,1]$ satisfying $\sum_{s' \in S} \delta(s,s') = 1$ for all $s \in S$, and a labeling function $\ell\colon S \rightarrow 2^{AP}$. The size $|\mathcal{M}|$ of $\mathcal{M}$ is defined as $\sum_{s,s' \in S} |\delta(s,s')|$, where $|p|$ denotes the length of the binary encoding of $p \in \mathbb{Q}$.

A path of $\mathcal{M}$ is an infinite sequence $\pi = s_0 s_1 s_2 \cdots \in S^\omega$ such that $\delta(s_n, s_{n+1}) > 0$ for all $n \in \mathbb{N}$. We say that π starts in s_0. For $n \in \mathbb{N}$, we write $\pi(n) = s_n$ for the n-th state of π and $\pi[n,\infty) = s_n s_{n+1} s_{n+2} \cdots$ for the suffix of π starting at position n. We write $\Pi(\mathcal{M}, s)$ for the set of all paths of $\mathcal{M}$ starting in $s \in S$.

The probability measure μ_s on sets of paths starting in some state $s \in S$ is defined as usual: Fix some non-empty path prefix $\rho = s_0 \cdots s_n$ starting in $s_0 = s \in S$. The probability of the cylinder set

$$C_\rho = \{\pi \in \Pi(\mathcal{M}, s) \mid \rho \text{ is a prefix of } \pi\}$$

is

$$\mu_s(C_\rho) = \prod_{j=0}^{n-1} \delta(\rho(j), \rho(j+1)),$$

i.e., the probability of the prefix ρ (which is 1 if $|\rho| = 1$, i.e., if $\rho = s$). Using Carathéodory's extension theorem, we lift μ_s to a measure on the σ-algebra induced by the cylinder sets of path prefixes starting in s. See, e.g., [12] for details. All sets of paths used in the following are measurable.

3 Robust PCTL

As the main concepts underlying robust semantics can be illustrated using the always and eventually operators, we will disregard the until and release operators in the following. They can be added straightforwardly to our new logics, but we refrain from doing so for the sake of brevity. Let PCTL($\Diamond, \Box$) denote the fragment of PCTL using the temporal operators eventually, always, and next only.

rPCTL and PCTL($\Diamond, \Box$) share the same syntax, i.e., the formulas of rPCTL are given by the grammar

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \rightarrow \varphi \mid \mathcal{P}_{\sim\lambda}(\odot\varphi) \mid \mathcal{P}_{\sim\lambda}(\boxdot\!\!\!\Diamond\varphi) \mid \mathcal{P}_{\sim\lambda}(\boxdot\varphi)$$

where p ranges over AP, $\sim \in \{<, \leq, =, \geq, >\}$, and $\lambda \in [0,1]$ is a rational probability threshold. The size $|\varphi|$ of a formula φ is defined as the number of subformulas of φ plus the maximal length $|\lambda|$ of the binary encodings of the thresholds $\lambda \in \mathbb{Q}$ appearing in φ.

The semantics of rPCTL is defined via an evaluation function $V_\mathcal{M}$ mapping a vertex s of a fixed DTMC $\mathcal{M} = (S, s_I, \delta, \ell)$ and a formula φ to a truth value in $\mathbb{B}_4 = \{1111, 0111, 0011, 0001, 0000\}$, which is ordered by

$$1111 \succ 0111 \succ 0011 \succ 0001 \succ 0000.$$

Given a truth value $t = b_1b_2b_3b_4 \in \mathbb{B}_4$, we write $t[k]$ for b_k.

The evaluation function is defined inductively via

- $V_{\mathcal{M}}(s, p) = \begin{cases} 1111 & \text{if } p \in \ell(s), \\ 0000 & \text{if } p \notin \ell(s), \end{cases}$
- $V_{\mathcal{M}}(s, \neg\varphi) = \begin{cases} 1111 & \text{if } V_{\mathcal{M}}(s, \varphi) \prec 1111, \\ 0000 & \text{if } V_{\mathcal{M}}(s, \varphi) = 1111, \end{cases}$
- $V_{\mathcal{M}}(s, \varphi_0 \wedge \varphi_1) = \min(V_{\mathcal{M}}(s, \varphi_0), V_{\mathcal{M}}(s, \varphi_1))$,
- $V_{\mathcal{M}}(s, \varphi_0 \vee \varphi_1) = \max(V_{\mathcal{M}}(s, \varphi_0), V_{\mathcal{M}}(s, \varphi_1))$,
- $V_{\mathcal{M}}(s, \varphi_0 \rightarrow \varphi_1) = \begin{cases} 1111 & \text{if } V_{\mathcal{M}}(s, \varphi_0) \preceq V_{\mathcal{M}}(s, \varphi_1), \\ V_{\mathcal{M}}(s, \varphi_1) & \text{if } V_{\mathcal{M}}(s, \varphi_0) \succ V_{\mathcal{M}}(s, \varphi_1), \end{cases}$
- $V_{\mathcal{M}}(s, \mathcal{P}_{\sim\lambda}(\odot\varphi)) = b_1b_2b_3b_4$ with $b_k = 1$ if and only if $\mu_s(\{\pi \in \Pi(\mathcal{M}, s) \mid V_{\mathcal{M}}(\pi(1), \varphi)[k] = 1\}) \sim \lambda$,
- $V_{\mathcal{M}}(s, \mathcal{P}_{\sim\lambda}(\diamondsuit\varphi)) = b_1b_2b_2b_4$ with $b_k = 1$ if and only if $\mu_s(\{\pi \in \Pi(\mathcal{M}, s) \mid (V_{\mathcal{M}}(\pi(n), \varphi))[k] = 1 \text{ for some } n \in \mathbb{N}\}) \sim \lambda$, and
- $V_{\mathcal{M}}(s, \mathcal{P}_{\sim\lambda}(\boxdot\varphi)) = b_1b_2b_3b_4$ with
  - $b_1 = 1$ if and only if $\mu_s(\{\pi \in \Pi(\mathcal{M}, s) \mid (V_{\mathcal{M}}(\pi(n), \varphi))[1] = 1 \text{ for all } n \in \mathbb{N}\}) \sim \lambda$,
  - $b_2 = 1$ if and only if $\mu_s(\{\pi \in \Pi(\mathcal{M}, s) \mid (V_{\mathcal{M}}(\pi(n), \varphi))[2] = 1 \text{ for all but finitely many } n \in \mathbb{N}\}) \sim \lambda$,
  - $b_3 = 1$ if and only if $\mu_s(\{\pi \in \Pi(\mathcal{M}, s) \mid (V_{\mathcal{M}}(\pi(n), \varphi))[3] = 1 \text{ for infinitely many } n \in \mathbb{N}\}) \sim \lambda$, and
  - $b_4 = 1$ if and only if $\mu_s(\{\pi \in \Pi(\mathcal{M}, s) \mid (V_{\mathcal{M}}(\pi(n), \varphi))[4] = 1 \text{ for some } n \in \mathbb{N}\}) \sim \lambda$.

For a detailed motivation and description of the semantics, we refer to [1].

Example 1. *Consider the formula $\varphi = \mathcal{P}_{\geq .9}(\boxdot a) \rightarrow \mathcal{P}_{\geq .95}(\boxdot g)$ expressing a robust assume-guarantee property. Assume φ evaluates to 1111 and consider the following cases:*

- *If $\mathcal{P}_{\geq .9}(\boxdot a)$ evaluates to 1111, i.e., with probability $\geq .9$, a holds at every position of a path. Then, by the semantics of the implication, with probability $\geq .95$, g holds at every position of a path.*
- *If $\mathcal{P}_{\geq .9}(\boxdot a)$ evaluates to 0111, i.e., with probability $\geq .9$, a holds at all but finitely many positions of a path (but not at* every *position of a path with probability $\geq .9$). Then, by the semantics of the implication, with probability $\geq .95$, g holds at least at all but finitely many positions of a path.*
- *Similar arguments hold for the truth values 0011 (with probability $\geq .9$, a holds infinitely often, which implies that, with probability $\geq .95$, g holds infinitely often) and 0001 (with probability $\geq .9$, a holds at least once, which implies that, with probability $\geq .95$, g holds at least once).*

Thus, the semantics of φ ensures that a violation of the assumption $\boxdot a$ is met with (at most) a proportional violation of the guarantee $\boxdot g$.

But we can even derive useful information if φ does not evaluate to 1111. Assume, φ evaluates to $t \prec 1111$. This can only be the case if the assumption $\mathcal{P}_{\geq .9}(\boxdot a)$ evaluates to some truth value strictly smaller than t and the guarantee $\mathcal{P}_{\geq .95}(\boxdot g)$ evaluates to t. Hence, even if the implication does not hold, it still yields the degree of satisfaction of the guarantee.

3.1 Expressiveness

In this section, we discuss the expressiveness of rPCTL; in particular, we compare it to the expressiveness of PCTL. More precisely, as we have defined rPCTL without the until and release operators, we also need to compare rPCTL to PCTL($\Diamond, \Box$), the fragment of PCTL allowing the next, eventually, and always operators, but not until and release. Nevertheless, the results we prove below also hold for the the full logics with until and release.

Our first result shows that rPCTL is at least as expressive as PCTL($\Diamond, \Box$). Note that the restriction to implication-free formulas is just technical, as implications $\varphi \rightarrow \psi$ in PCTL formulas can always be rewritten as $\neg\varphi \vee \psi$. The need for the implication-removal stems from the fact that robust implication does not generalize classical implication (see Footnote 3 of [7]).

Lemma 1. *Let φ be a formula of PCTL($\Diamond, \Box$) without implications, and let $\mathcal{M}$ be a DTMC with initial state s_I. Then, $\mathcal{M}, s_I \models \varphi$ if and only if $V_{\mathcal{M}}(s_I, \dot{\varphi}) = 1111$, where $\dot{\varphi}$ is the rPCTL formula obtained from φ by dotting all temporal operators.*

Proof. By induction over the construction of φ. □

Corollary 1. *rPCTL is at least as expressive as PCTL.*

Let us briefly discuss the other inclusion, e.g., is rPCTL strictly more expressive than PCTL? This is true for the nonprobabilistic setting, where rCTL (robust CTL) is strictly more expressive than CTL, as $V_{\mathcal{M}}(s_I, \forall \boxdot p) \succeq 0011$ holds if and only if p holds infinitely often on every path starting in s. This property cannot be expressed in CTL [3]. However, the analogous property "p holds infinitely often with probability one" can be expressed in PCTL [3] (say, when considering finite DTMC's), relying on the fact that a path ends up with probability one in a bottom strongly-connected component. We leave open the question whether similar arguments are sufficient to show that rPCTL can be embedded into PCTL (w.r.t. finite DTMC's).

Let us conclude this section with a consequence of the embedding proven in Lemma 1. rPCTL satisfiability asks, given a formula φ and a truth value t^* whether there is a DTMC $\mathcal{M}$ with initial state s_I such that $V_{\mathcal{M}}(s_I, \varphi) \succeq t^*$. Decidability of PCTL satisfiability is an open problem [5] (to the best of our knowledge even in the setting without until and release considered here). So, due to Lemma 1, which allows us to embed PCTL in rPCTL, settling the decidability of rPCTL satisfiability problem is most likely challenging.

3.2 Model-checking

In this section, we prove that model-checking rPCTL is not harder than model-checking PCTL, which is in PTime [6], i.e., robustness can be added for free. Formally, rPCTL model-checking is the following problem: Given a DTMC $\mathcal{M}$ with initial state s_I, an rPCTL formula φ, and a truth value $t^* \in \mathbb{B}_4$, is $V_{\mathcal{M}}(s_I, \varphi) \succeq t^*$?

Theorem 1. *rPCTL model-checking is in* PTime.

Proof. Fix $\mathcal{M} = (S, s_I, \delta, \ell)$ and let $\mathrm{cl}(\varphi)$ denote the set of subformulas of φ (which is defined as expected). We show how to inductively compute the satisfaction sets

$$\mathrm{Sat}(\psi, t) = \{s \in S \mid V_{\mathcal{M}}(s, \psi) \succeq t\}$$

for $\psi \in \mathrm{cl}(\varphi)$ and $t \in \mathbb{B}_4$. Note that $\mathrm{Sat}(\psi, 0000) = S$ holds for all subformulas ψ. Hence, in the following, we only consider $t \succ 0000$. Also, the cases for atomic propositions and Boolean connectives are trivial. Hence, we only need to consider ψ of the form $\mathcal{P}_{\sim\lambda}(\odot\,\psi')$, $\mathcal{P}_{\sim\lambda}(\dot{\Diamond}\,\psi')$, or $\mathcal{P}_{\sim\lambda}(\boxdot\,\psi')$.

We begin with the case of the robust next operator. We have $s \in \mathrm{Sat}(\mathcal{P}_{\sim\lambda}(\odot\,\psi'), t)$ if and only if

$$\mu_s(\{\pi \in \Pi(\mathcal{M}, s) \mid \pi(1) \in \mathrm{Sat}(\psi', t)\}) = \left(\sum_{s' \in \mathrm{Sat}(\psi', t)} \delta(s, s') \right) \sim \lambda.$$

The value $\sum_{s'} \delta(s, s')$ can be computed and compared to λ in polynomial time, as $\mathrm{Sat}(\psi, t)$ has already been computed by induction hypothesis.

It remains to consider the cases of the robust eventually and the robust always operator. It will turn out that the former is a special case of the latter, which we therefore consider first.

Note that we have $s \in \mathrm{Sat}(\mathcal{P}_{\sim\lambda}(\boxdot\, \psi'), t)$ if and only if

- $t = 1111$ and $\mu_s(\mathrm{Safe}(s, \mathrm{Sat}(\psi', 1111))) \sim \lambda$,
- $t = 0111$ and $\mu_s(\mathrm{CoBüchi}(s, \mathrm{Sat}(\psi', 0111))) \sim \lambda$,
- $t = 0011$ and $\mu_s(\mathrm{Büchi}(s, \mathrm{Sat}(\psi', 0011))) \sim \lambda$, and
- $t = 0001$ and $\mu_s(\mathrm{Reach}(s, \mathrm{Sat}(\psi', 0001))) \sim \lambda$,

where

- $\mathrm{Safe}(s, S') = \{\pi \in \Pi(\mathcal{M}, s) \mid \pi(n) \in S' \text{ for all } n \in \mathbb{N}\}$,
- $\mathrm{CoBüchi}(s, S') = \{\pi \in \Pi(\mathcal{M}, s) \mid \pi(n) \in S' \text{ for all but finitely many } n \in \mathbb{N}\}$,
- $\mathrm{Büchi}(s, S') = \{\pi \in \Pi(\mathcal{M}, s) \mid \pi(n) \in S' \text{ for infinitely many } n \in \mathbb{N}\}$, and
- $\mathrm{Reach}(s, S') = \{\pi \in \Pi(\mathcal{M}, s) \mid \pi(n) \in S' \text{ for some } n \in \mathbb{N}\}$.

As the satisfiability sets $\mathrm{Sat}(\psi', t)$ are already computed by induction assumption, we only need to compute whether the probability of some safety, coBüchi, Büchi, or reachability condition is (strictly) larger, equal, or (strictly) smaller than a given threshold. This can be achieved in polynomial time [4].

Finally, $s \in \mathrm{Sat}(\mathcal{P}_{\sim\lambda}(\diamondsuit\, \psi'), t)$ holds if and only if $\mu_s(\mathrm{Reach}(s, \mathrm{Sat}(\psi', t))) \sim \lambda$, which we have just seen how to check.

Altogether, our algorithm inductively computes $5|\mathrm{cl}(\varphi)|$ many satisfaction sets, each one in polynomial time, and then checks whether $s_I \in \mathrm{Sat}(\varphi, t^*)$. Thus, the algorithm has polynomial-time running time. □

4 Robust PCTL*

In this section, we robustify *PCTL∗*. Again, we disregard until and release: As for rPCTL, all our results below for rPCTL* also hold for the full logic with until and release. Let PCTL*($\diamondsuit, \square$) denote the fragment of PCTL* using only the temporal operators always, eventually, and next.

rPCTL* and PCTL*($\diamondsuit, \square$) share the same syntax, i.e., the formulas of rPCTL are either state formulas or path formulas. State formulas are given by the grammar

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \rightarrow \varphi \mid \mathcal{P}_{\sim\lambda}(\Phi)$$

where p ranges over AP, $\sim \in \{<, \leq, =, \geq, >\}$, and $\lambda \in [0, 1]$ is a probability threshold, and Φ ranges over path formulas. Paths formulas are given by

$$\Phi ::= \varphi \mid \neg\Phi \mid \Phi \wedge \Phi \mid \Phi \vee \Phi \mid \Phi \rightarrow \Phi \mid \odot\,\Phi \mid \diamondsuit\,\Phi \mid \boxdot\,\Phi$$

where φ ranges over state formulas. Formula size is defined as for rPCTL.

The semantics of rPCTL* is again defined via an evaluation function $V_{\mathcal{M}}$, this time mapping a vertex s of a fixed DTMC $\mathcal{M} = (S, s_I, \delta, \ell)$ and a state formula or a path of $\mathcal{M}$ and a path formula to a truth value in $\mathbb{B}_4$. The evaluation function is defined inductively via

- $V_{\mathcal{M}}(s, p) = \begin{cases} 1111 & \text{if } p \in \ell(s), \\ 0000 & \text{if } p \notin \ell(s), \end{cases}$
- $V_{\mathcal{M}}(s, \neg\varphi) = \begin{cases} 1111 & \text{if } V_{\mathcal{M}}(s, \varphi) \prec 1111, \\ 0000 & \text{if } V_{\mathcal{M}}(s, \varphi) = 1111, \end{cases}$
- $V_{\mathcal{M}}(s, \varphi_0 \wedge \varphi_1) = \min(V_{\mathcal{M}}(s, \varphi_0), V_{\mathcal{M}}(s, \varphi_1))$,
- $V_{\mathcal{M}}(s, \varphi_0 \vee \varphi_1) = \max(V_{\mathcal{M}}(s, \varphi_0), V_{\mathcal{M}}(s, \varphi_1))$,

- $V_\mathcal{M}(s, \varphi_0 \rightarrow \varphi_1) = \begin{cases} 1111 & \text{if } V_\mathcal{M}(s,\varphi_0) \preceq V_\mathcal{M}(s,\varphi_1), \\ V_\mathcal{M}(s,\varphi_1) & \text{if } V_\mathcal{M}(s,\varphi_0) \succ V_\mathcal{M}(s,\varphi_1), \end{cases}$
- $V_\mathcal{M}(s, \mathcal{P}_{\sim\lambda}(\Phi) = \max\{b \in \mathbb{B}_4 \mid \mu_s(\{\pi \in \Pi(\mathcal{M}, s) \mid V_\mathcal{M}(\pi, \Phi) \geq b\}) \sim \lambda\}$,
- $V_\mathcal{M}(\pi, \varphi) = V_\mathcal{M}(\pi(0), \varphi)$,
- $V_\mathcal{M}(\pi, \neg\Phi) = \begin{cases} 1111 & \text{if } V_\mathcal{M}(\pi,\Phi) \prec 1111, \\ 0000 & \text{if } V_\mathcal{M}(\pi,\Phi) = 1111, \end{cases}$
- $V_\mathcal{M}(\pi, \Phi_0 \wedge \Phi_1) = \min(V_\mathcal{M}(\pi, \Phi_0), V_\mathcal{M}(\pi, \Phi_1))$,
- $V_\mathcal{M}(\pi, \Phi_0 \vee \Phi_1) = \max(V_\mathcal{M}(\pi, \Phi_0), V_\mathcal{M}(\pi, \Phi_1))$,
- $V_\mathcal{M}(\pi, \Phi_0 \rightarrow \Phi_1) = \begin{cases} 1111 & \text{if } V_\mathcal{M}(\pi,\Phi_0) \preceq V_\mathcal{M}(\pi,\Phi_1), \\ V_\mathcal{M}(s,\Phi_1) & \text{if } V_\mathcal{M}(\pi,\Phi_0) \succ V_\mathcal{M}(\pi,\Phi_1), \end{cases}$
- $V_\mathcal{M}(\pi, \odot\Phi) = V_\mathcal{M}(\pi[1,\infty), \Phi)$,
- $V_\mathcal{M}(\pi, \dot\Diamond\Phi) = b_1b_2b_3b_4$ with $b_k = \max_{n\geq 0}(V_\mathcal{M}(\pi[n,\infty), \Phi))[k]$, and
- $V_\mathcal{M}(\pi, \boxdot\Phi) = b_1b_2b_3b_4$ with
  - $b_1 = \min_{n\geq 0}(V_\mathcal{M}(\pi[n,\infty), \Phi))[1]$,
  - $b_2 = \max_{m\geq 0}(\min_{n\geq m} V_\mathcal{M}(\pi[n,\infty), \Phi))[2]$,
  - $b_3 = \min_{m\geq 0}(\max_{n\geq m} V_\mathcal{M}(\pi[n,\infty), \Phi))[3]$, and
  - $b_4 = \max_{n\geq 0}(V_\mathcal{M}(\pi[n,\infty), \Phi))[4]$.

Note how the semantics of Boolean operators in state and path formulas is similar, and that the semantics generalizes the semantics of rPCTL. Finally, the semantics of path formulas are derived from the (nonprobabilistic) semantics of rLTL [13].

Example 2. *Consider the formula $\mathcal{P}_{\geq .9}(\boxdot a \rightarrow \boxdot g)$, a variant of the assume-guarantee property of Example 1. It evaluates to the largest truth value t such that $\boxdot a \rightarrow \boxdot g$ evaluates to t with probability $\geq .9$. Now, on a single path, $\boxdot a \rightarrow \boxdot g$ evaluates to*

- 1111 *if $\boxdot g$ evaluates to a larger or equal truth value than $\boxdot a$ and*
- *to $t \prec$ 1111 if $\boxdot a$ evaluates to t and $\boxdot g$ evaluates to a truth value larger than t.*

4.1 Expressiveness

As usual for temporal logics that allow arbitrary nesting of temporal operators (without the need for path quantifiers in between like in CTL-style logics), rPCTL* has the same expressiveness as its nonrobust version. Note that we again compare, for the sake of simplicity, to PCTL*($\Diamond, \Box$), the fragment of PCTL* without until and release, but our results can easily be extended to full PCTL* with until and release.

Theorem 2. *rPCTL* is as expressive as PCTL*($\Diamond, \Box$). Both translations can be computed in polynomial time.*

Proof. The translation from PCTL*($\Diamond, \Box$) to rPCTL* is a generalization of the analogous result for PCTL and rPCTL (see Lemma 1): Let φ be a PCTL*($\Diamond, \Box$) state formula without implications, and let $\mathcal{M}$ be a DTMC with initial state s_I. Then, $\mathcal{M}, s_I \models \varphi$ if and only if $V_\mathcal{M}(s_I, \dot\varphi) = 1111$, where $\dot\varphi$ is the rPCTL* state formula obtained from φ by dotting all temporal operators. This is again proven by induction over the construction of φ.

For the other direction, we inductively translate an rPCTL* state formula φ and a truth value $t \in \mathbb{B}_4$ into a PCTL* state formula φ_t such that $V_\mathcal{M}(s, \varphi) \succeq t$ if and only if $\mathcal{M}, s \models \varphi_t$ all DTMC's $\mathcal{M}$ and all states s of $\mathcal{M}$.

Note that we have $V_\mathcal{M}(s, \varphi) \succeq 0000$ for all state formulas φ. Hence, we can define φ_{0000} to be some tautology (say $p \vee \neg p$) and only consider $t \succ 0000$ in the following. We start with atomic propositions and define

- $p_t = p$,

The translation for Boolean operators is the same for state and path formulas. Hence, to avoid duplication, φ ranges in the following both over state and path formulas.

- $(\neg\varphi)_t = \neg\varphi_t$,
- $(\varphi_1 \vee \varphi_2)_t = (\varphi_1)_t \vee (\varphi_2)_t$,
- $(\varphi_1 \wedge \varphi_2)_t = (\varphi_1)_t \wedge (\varphi_2)_t$,
- $(\varphi_1 \rightarrow \varphi_2)_{1111} = \bigwedge_{t \succeq 0000} (\varphi_2)_t \vee \neg(\varphi_1)_t$, and
- $(\varphi_1 \rightarrow \varphi_2)_t = (\varphi_1 \rightarrow \varphi_2)_{1111} \vee (\varphi_2)_t$ for $t \prec 1111$.

Finally, we consider the remaining operators.

- $(\mathcal{P}_{\sim\lambda}(\Phi))_t = \mathcal{P}_{\sim\lambda}(\Phi_t)$,
- $(\odot\,\Phi)_t = \bigcirc\,\Phi_t$,
- $(\diamond\!\!\!\cdot\;\Phi)_t = \Diamond\,\Phi_t$, and
- $(\boxdot\,\Phi)_{1111} = \Box\,\Phi_{1111}$,
- $(\boxdot\,\Phi)_{0111} = \Diamond\Box\,\Phi_{0111}$,
- $(\boxdot\,\Phi)_{0011} = \Box\Diamond\,\Phi_{0011}$, and
- $(\boxdot\,\Phi)_{0001} = \Diamond\,\Phi_{0001}$.

An induction over the construction of φ shows that φ_t has the desired properties. □

4.2 Model-checking

The model-checking problem for rPCTL* is defined as for rPCTL: Given a DTMC $\mathcal{M}$ with initial state s_I, an rPCTL* state formula φ, and a truth value $t^* \in \mathbb{B}_4$, is $V_{\mathcal{M}}(s, \varphi) \succeq t^*$? It is PSPACE-complete, as is the PCTL* model-checking problem [2, 14], i.e., robustness comes again for free.

Theorem 3. *rPCTL* model-checking is* PSPACE*-complete.*

Proof. The result follows immediately from Theorem 2 and the PSPACE-completeness of PCTL* model-checking. □

5 Conclusion

We have shown how to robustify PCTL and PCTL*, obtaining the logics rPCTL and rPCTL*. The model-checking problems for these robust logics are as hard as the model-checking problems for the nonrobust variants, i.e., robustness can be added for free. This is in line with previous work on robust variants of LTL [1] and its extensions [11], as well as CTL and CTL* [8]. Probably the most interesting problem left for future work concerns the expressiveness of rPCTL and PCTL. Note that in the nonprobabilistic setting, it is known that rCTL is strictly more expressive than CTL [8]. However, as discussed in Subsection 3.1, it is unclear whether this separation can be lifted to the probabilistic setting.

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