Riesz Modal Logic for Markov Processes

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***Abstract*—We investigate a modal logic for expressing properties of Markov processes whose semantics is real-valued, rather than Boolean, and based on the mathematical theory of Riesz spaces. We use the duality theory of Riesz spaces to provide a connection between Markov processes and the logic. This takes the form of a duality between the category of coalgebras of the Radon monad (modeling Markov processes) and the category of a new class of algebras (algebraizing the logic) which we call modal Riesz spaces. As a result, we obtain a sound and complete axiomatization of the Riesz Modal logic.**

I. Introduction

Directed graphs and similar structures, such as labeled transition systems and Kripke frames, are the mathematical objects used to represent, by means of operational semantics, the behavior of (nondeterministic) computer programs. For this reason, a large body of research has focused on the study of logics for expressing useful properties of directed graphs. Among these, basic *modal logic* and its extensions (e.g., CTL, modal μ-calculus, *etc.*) play a fundamental role (see, e.g., [1]).

Despite their wide applicability, directed graphs are not adequate for modelling probabilistic programs such as those involving commands for generating random numbers (e.g., `x=rand()` in `C++`). These programs are naturally modelled by Markov chains or similar structures such as (labelled) Markov decision processes (see, e.g., [2]). Consequently, a number of logics for expressing properties of Markov chains have been investigated with pCTL ([3], [4]) being arguably the most studied. Despite efforts spanning over three decades, however, the theory of pCTL is not yet well understood. For example no sound and complete axiomatizations have been found and the decidability of the satisfiability problem is open in the literature ([5], [4]).

In an attempt to make some progress, some research has focused on *real–valued* logics for expressing properties of Markov chains and similar systems. Following Kozen's seminal work [6], formulas ϕ of these logics are interpreted as real–valued functions $[\![\phi]\!]: X \to \mathbb{R}$ on the state space X rather than (characteristic functions of) sets $[\![\phi]\!] : X \to \{0, 1\}$ as in ordinary Boolean logics such as pCTL. Importantly, it has recently been shown ([7], [8], see also [9]) that the *Łukasiewicz modal μ–calculus*, obtained by enriching a basic real–valued modal logic with fixed-point operators (as in the modal μ-calculus [10]), is sufficiently expressive to interpret pCTL. Hence the quantitative approach to modal logics for Markov processes suffices to express most properties of interest. On the other hand, this also implies that axiomatizing the Łukasiewicz modal μ–calculus is a challenging problem.

In this work we consider the problem of axiomatizing a basic real–valued modal logic for Markov processes called *Riesz Modal logic*. Importantly, this logic subsumes other real–valued modal logics appeared in the literature, such as the logic considered in [11, §8.2] to characterize behavioral metrics and the fixed-point free fragment of the Łukasiewicz modal μ–calculus. Hence this work can be understood as a first fundamental step towards the axiomatization of the full Łukasiewicz modal μ–calculus (and therefore pCTL).

The (interpretations of the) connectives of the Riesz Modal logic are carefully chosen to be the basic operations of *Riesz spaces*, also known as *lattice-ordered lattices* or simply *vector lattices* [12] . These are the vector space operations of addition ($+$) and multiplication by real scalars ($r\cdot$) and the lattice operations of meet ($\sqcap$) and join ($\sqcup$). Historically, research on Riesz spaces was pioneered in the 1930's by F. Riesz, L. Kantorovich and H. Freudenthal among others and was motivated by the applications in the study function spaces ($X \to \mathbb{R}$) in functional analysis. In this work we exploit the rich *duality theory* of Riesz spaces to develop a theory of duality for Markov processes and the Riesz modal logic. This is the analogue of the Stone duality theory for Kripke frames and ordinary modal logic based on the algebraic notion of *Boolean algebra with operators* (or modal Boolean algebras) [13]. Specifically, we introduce the notion of modal Riesz spaces, which are just Riesz spaces endowed with a unary operation $\Diamond: R \to R$ satisfying certain axioms. We then exhibit a duality between the category of modal Riesz spaces and the category of coalgebras of the Radon monad in the category of Compact Hausdorff spaces (Theorem V.1). Such coalgebras can be seen as *topological Markov processes*. The desired axiomatization of the Riesz modal logic is obtained as a direct consequence of this general duality theorem.

The theory of duality is considered by van Benthem as one of the "three pillars of wisdom" in the edifice of standard modal logic [14]. Accordingly, we argue that our result is a fundamental first step in the development of a theory of real–valued logics for probabilistic systems.

Related work. In [15] the authors develop a duality theory for standard modal logic and coalgebras of the *Vietoris functor* (*i.e.*, topological Kripke frames) based on Stone duality for

Boolean algebras. Our work is conceptually very similar to that of [15] as we consider topological Markov processes in place of topological Kripke frames and use the duality theory of Riesz spaces in place of that of Boolean algebras.

As mentioned earlier, the study of real–valued logics for probabilistic systems was pioneered by Kozen [6] in his seminal work on the logic *probabilistic PDL* (pPDL). Many of the ideas underlying our work are directly inspired from [6]. A difference is that, using Pnueli's terminology [16], the logic pPDL is *exogenous* while the Riesz modal logic, pCTL and the Łukasiewicz modal μ–calculus are *endogenous*. In endogenous logics the language of properties (logical formulas) is independent of the language of programs (process terms).

Another related work is [17] where a theory of Markov processes is developed using the duality theory of commutative C*-algebras. Informally, this approach takes multiplication of reals as a basic operation instead of the lattice operations of meet and join, as in the theory of Riesz spaces. Since the lattice operations are used in all real–valued logics for probabilistic systems we are aware of, the theory of Riesz spaces is arguably simpler to handle for studying such logics.

Organization of the paper. In Section II we collect the required mathematical background on probability measures on topological spaces, Markov processes as coalgebras and the theory of Riesz spaces. This section is rather long but hopefully serves the purpose of making this work sufficiently self contained. In Section III we introduce the syntax and semantics of the Riesz Modal logic. In Section IV we introduce the notion of modal Riesz spaces and in Section V we state our main duality theorem. Lastly, in sections VI and VII we apply the duality theorem to obtain some consequences including the axiomatization of the Riesz modal logic.

II. Technical background

A. Topology, measures and Riesz–Markov–Kakutani representation theorem

We denote by **CHaus** the category of compact Hausdorff spaces with continuous maps as morphisms. If X is a compact Hausdorff space, we denote with $\mathcal{B}(X)$ the collection of Borel sets of X, *i.e.*, the smallest σ-algebra of subsets of X containing all open sets. A *(Borel) subprobability measure* on X is a function $\mathcal{B}(X) \to [0,1]$ such that $\mu(\emptyset)=0$, $\mu(X)\leq 1$ and $\mu(\bigcup_n A_n)=\sum_n \mu(A_n)$ for all countable sequences (A_n) of pairwise disjoint Borel sets. The measure μ is a *probability measure* if $\mu(X)=1$.

Definition II.1 (Radon probability measure)**.** A subprobability measure μ on the compact Hausdorff space X is *Radon* if for every Borel set A, $\mu(A) = \sup\{\ \mu(K) \mid K \subseteq A \text{ and } K \text{ is compact}\}$.

In other words, a probability measure is Radon if the probability $\mu(A)$ of every Borel set A can be approximated to any degree of precision by compact subsets of A. Most naturally occurring probability (sub–)measures are Radon. In particular, if X is a Polish space, all (sub–)probability measures are Radon.

Given a set X, we denote the collection of all functions $X \to \mathbb{R}$ by $\mathbb{R}^X$. If X is a compact Hausdorff space, then $C(X)$ denotes the subset of $\mathbb{R}^X$ consisting of all continuous functions. We use $\mathbb{0}_X$ and $\mathbb{1}_X$ to denote the constant (continuous) functions defined as $\mathbb{0}_X(x) = 0$ and $\mathbb{1}_X(x) = 1$, for all $x \in X$, respectively. Using the vector space operations of $\mathbb{R}$ pointwise, both $\mathbb{R}^X$ and $C(X)$ can be given the structure of a $\mathbb{R}$-vector space. Furthermore, the ordering ($\leq$) defined pointwise as $f \leq g \Leftrightarrow \forall x. f(x) \leq g(x)$ is a lattice on both $\mathbb{R}^X$ and $C(X)$.

Given a compact Hausdorff space X and a (sub–)probability measure μ on X, one can define the expectation functional $\mathbb{E}_\mu : C(X) \to \mathbb{R}$ as

$$\mathbb{E}_\mu(f) = \int_X f \,\mathrm{d}\mu \tag{1}$$

where the integral is well defined because any $f \in C(X)$, being continuous and defined on a compact space, is measurable and bounded. One can then observe that: (i)

1) $\mathbb{E}_\mu$ is a *linear* map: $\mathbb{E}_\mu(f_1 + f_2) = \mathbb{E}_\mu(f_1) + \mathbb{E}_\mu(f_2)$, and $\mathbb{E}_\mu(rf) = r\mathbb{E}_\mu(f)$, for all $r \in \mathbb{R}$,
2) $\mathbb{E}_\mu$ is *positive*: $\mathbb{E}_\mu(f) \geq 0$ if $f \geq \mathbb{0}_X$, and
3) $\mathbb{E}_\mu$ is $\mathbb{1}_X$*-decreasing*: $\mathbb{E}_\mu(\mathbb{1}_X) \leq 1$.

The latter inequality becomes an equality if μ is a probability measure.

The celebrated Riesz–Markov–Kakutani representation theorem states that in fact any such functional corresponds to a unique Radon subprobability measure.

Theorem II.2 (Riesz–Markov–Kakutani)**.** *Let X be a compact Hausdorff space. For every functional $F : C(X) \to \mathbb{R}$ such that (i) F is linear, (ii) F is positive and (iii) $F(\mathbb{1}_X) \leq 1$, there exists one and only one Radon subprobability measure μ on X such that $F = \mathbb{E}_\mu$.*

Given a compact Hausdorff space X we denote with $\mathcal{R}^{\leq 1}(X)$ the collection of all Radon subprobability measures on X. Equivalently, by the Riesz–Markov–Kakutani theorem, we can identify $\mathcal{R}^{\leq 1}(X)$ with the collection of functionals

$$\big\{F : C(X) \to \mathbb{R} \mid F \text{ is linear, positive and } \mathbb{1}_X\text{-decreasing}\big\}.$$

The set $\mathcal{R}^{\leq 1}(X)$ can be endowed with the weak-* topology, the coarsest (*i.e.*, having fewest open sets) topology such that, for all $f \in C(X)$, the map $T_f : \mathcal{R}^{\leq 1}(X) \to \mathbb{R}$, defined as $T_f(F) = F(f)$, is continuous. The weak-* topology on $\mathcal{R}^{\leq 1}(X)$ is compact and Hausdorff by the Banach-Alaoglu theorem. Hence $\mathcal{R}^{\leq 1}$ maps a compact Hausdorff space X to the compact Hausdorff space $\mathcal{R}^{\leq 1}(X)$. In fact $\mathcal{R}^{\leq 1}$ becomes a functor on **CHaus** by defining, for any continuous map $f : X \to Y$ in **CHaus**, the continuous map $\mathcal{R}^{\leq 1}(f) : \mathcal{R}^{\leq 1}(X) \to \mathcal{R}^{\leq 1}(Y)$ as

$$\mathcal{R}^{\leq 1}(f)(F)(g) = F(g \circ f), \tag{2}$$

for all $g \in C(Y)$.

The functor $\mathcal{R}^{\leq 1}$ is shown to be the underlying functor of a monad in [18, §6], based on the previous work of Świrszcz

[19], [20] (see also Giry's work [21]). However, we will not require the monad structure for the purposes of this article. Following [22], we call $\mathcal{R}^{\leq 1}$ the *Radon monad*.

B. Markov Processes and Coalgebra

Informally, a (discrete-time) Markov process consists of a set of states X and a transition function τ that associates to each state $x \in X$ a probability distribution $\tau(x)$ on the state space X. This mathematical object is interpreted, given an initial state x_0, as generating an infinite trajectory (or "computation") $(x_n)_{n\in\mathbb{N}}$ in the state space X, where x_{n+1} is chosen randomly using the probability distribution $\tau(x_n)$. A slight variant of this model, allowing the generation of infinite as well as finite trajectories, uses transition functions τ associating to each state x a subprobability distribution $\tau(x)$. The intended interpretation is that the computation will stop at state x with probability $1 - m_x$, where $m_x \in [0, 1]$ is the total mass of $\tau(x)$, and will continue with probability m_x following the (normalized) probability distribution $\tau(x)$.

This informal description readily translates to a formal definition for Markov processes having finite or countably infinite state space X, also known as *Markov chains*. When X is uncountable, some technical assumptions must be considered. Typically, X is assumed to be a topological or measurable space and τ is defined as a map from X to the collection of (sub–)probability measures on X satisfying certain convenient regularity assumption.

In this work we formally Markov define process as follows.

Definition II.3. A *(discrete-time) Markov process* is a pair (X, τ) such that X is a compact Hausdorff topological space and $\tau : X \to \mathcal{R}^{\leq 1}(X)$ is a continuous map.

Remark II.4. Many other variants can be considered. For example Markov processes having *labeled transitions* as in [11, §8.2] or endowed with a collection of predicates over the state space, *etc.* The results presented in this work can be smoothly adapted to these settings as well. See also the final discussion in Section VIII.

The following examples are particularly relevant in computer science.

Example II.5 (Finite Markov chains). Every finite Markov chain is a Markov process. Indeed, when the finite state space X is given the discrete topology, the space $\mathcal{R}^{\leq 1}(X)$ is isomorphic with the set $\mathcal{D}^{\leq 1}(X) = \{d : X \to [0, 1] \mid \sum_x d(x) \leq 1\}$ of all subprobability distributions on X and any function $\tau : X \to \mathcal{R}^{\leq 1}(X)$ is continuous.

Finite Markov chains are of central importance but, interestingly, logics such as pCTL are known to have formulas satisfiable by infinite Markov chains but not satisfiable by any finite Markov chain [5]. The following example shows how one can address this fact within our framework.

Example II.6 (Infinite Markov chains). Suppose we wish to model a Markov chain (X, τ) having an infinite discrete state space X and transition function $\tau : X \to \mathcal{D}^{\leq 1}(X)$. The set X endowed with the discrete topology is Hausdorff. Since X is discrete, the space $\mathcal{R}^{\leq 1}(X)$ of Radon subprobability measures on X is isomorphic with $\mathcal{D}^{\leq 1}(X)$ and any function $\tau : X \to \mathcal{R}^{\leq 1}(X)$ is continuous. But (X, τ) is not a Markov process in the sense of Definition II.3 because X is not compact. However we can consider the Stone–Čech compactification $\hat{X}$ of X, which is a compact Hausdorff space in which X embeds as a dense subset. Since $X \subseteq \hat{X}$, every measure $\mu \in \mathcal{R}^{\leq 1}(X)$ is the restriction of some measure $\hat{\mu} \in \mathcal{R}^{\leq 1}(\hat{X})$. Hence we can view the transition map τ as having type $\tau : X \to \mathcal{R}^{\leq 1}(\hat{X})$ and τ is continuous because X is discrete. By the universal property of the Stone–Čech compactification, there exists a unique continuous map $\hat{\tau} : \hat{X} \to \mathcal{R}^{\leq 1}(\hat{X})$ extending τ to the space $\hat{X}$. Therefore $(\hat{X}, \hat{\tau})$ is a Markov process embedding the original Markov chain (X, τ).

The example above shows that, although not all Markov chains can be directly modeled as Markov processes (due to the compactness requirement on the state space), the class of Markov processes is sufficiently large for reasoning about satisfiability of formulas (in the several logics for expressing properties of Markov chains) with respect to arbitrary Markov chains: if a formula ϕ is satisfied by some state $x \in X$ of a Markov chain (X, τ) then it is also satisfied[1] by the state $x \in \hat{X}$ of the Markov process $(\hat{X}, \hat{\tau})$.

The theory of coalgebra (for a comprehensive introduction see [23]) provides a convenient framework for formalizing the notion of morphism between Markov processes. The following is an equivalent reformulation of Definition II.3 in coalgebraic terms and relies on the fact, discussed earlier, that $\mathcal{R}^{\leq 1}$ is an endofunctor on the category $\mathbf{CHaus}$.

Definition II.7. A *discrete-time Markov process* is a coalgebra of the endofunctor $\mathcal{R}^{\leq 1}$ in the category $\mathbf{CHaus}$, *i.e.*, it is a morphism $\alpha : X \to \mathcal{R}^{\leq 1}(X)$ in $\mathbf{CHaus}$. A *(coalgebra) morphism* between the coalgebra $\alpha : X \to \mathcal{R}^{\leq 1}(X)$ and the coalgebra $\beta : Y \to \mathcal{R}^{\leq 1}(Y)$ is a continuous function $f : X \to Y$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "f"'] & \mathcal{R}^{\leq 1}(X) \arrow[d, "\mathcal{R}^{\leq 1}(f)"] \\ Y \arrow[r, "\beta"'] & \mathcal{R}^{\leq 1}(Y). \end{tikzcd} \tag{3}$$

Such a morphism will be denoted by $\alpha \xrightarrow{f} \beta$.

Definition II.8 (Category of Markov Processes)**.** We define the category $\mathbf{Markov}$ of Markov processes to be $\mathbf{CoAlg}(\mathcal{R}^{\leq 1})$ where the objects are coalgebras $\alpha : X \to \mathcal{R}^{\leq 1}(X)$ in $\mathbf{CHaus}$ and morphisms $\alpha \xrightarrow{f} \beta$ are coalgebra morphisms.

It is a well known fact that $\mathbf{CoAlg}(F)$ is always a category, for any functor F. In computer science, and in particular in the field of semantics and concurrency theory, one specific coalgebra in $\mathbf{CoAlg}(F)$ plays an important role. This is (when

[1]Of course in order to make this statement precise, one first needs to extend the standard evaluation of (e.g., pCTL) formulas from discrete Markov chains to arbitrary Markov processes. This can be done in the expected way.

it exists) the final object $\alpha : X \to F(X)$ in $\mathbf{CoAlg}(F)$, and is called the *final coalgebra*. The universal property that characterizes α is that, for every other F-coalgebra $\beta : Y \to F(Y)$, there exists one and only one coalgebra morphism $\beta \xrightarrow{\eta} \alpha$ in $\mathbf{CoAlg}(F)$. This property allows to interpret the domain X of α as the space of all "behaviours" as follows: given any coalgebra $\beta : Y \to \mathcal{R}^{\leq 1}(Y)$, the behaviour of the state y is the point $\eta(y) \in X$. And two states $y_1, y_2 \in Y$ are "behaviourally equivalent" if $\eta(y_1) = \eta(y_2)$.

For this reason in Section VI we study the final Markov process, *i.e.*, the final object in $\mathbf{Markov}$.

C. Riesz Spaces

This section contains the basic definitions and results related to Riesz spaces. We refer to [12] for a comprehensive reference to the subject.

A Riesz space is an algebraic structure $(A, 0, +, (r)_{r\in\mathbb{R}}, \sqcup, \sqcap)$ such that $(A, 0, +, (r)_{r\in\mathbb{R}})$ is a vector space over the reals, $(A, \sqcup, \sqcap)$ is a lattice and the induced order ($a \leq b \Leftrightarrow a \sqcap b = a$) is compatible with addition in the sense that: (i) for all $a, b, c \in A$, if $a \leq b$ then $a + c \leq b + c$, and (ii) if $a \geq 0$ and $r \in \mathbb{R}_{\geq 0}$ is a positive real, then $ra \geq 0$. Formally we have:

Definition II.9 (Riesz Space)**.** The *language* $\mathcal{L}_R$ of Riesz spaces is given by the (uncountable) signature $\{0, +, (r)_{r\in\mathbb{R}}, \sqcup, \sqcap\}$ where 0 is a constant, $+$, $\sqcup$ and $\sqcap$ are binary functions and r is a unary function, for all $r \in \mathbb{R}$. A *Riesz space* is a $\mathcal{L}_R$-algebra satisfying the following equations:

1) axioms of $\mathbb{R}$-vector spaces: $x + (y + z) = (x + y) + z$, $x + y = y + x$, $x + 0 = x$, $x + (-x) = 0$, $1x = x$, $r(r'x) = (r \cdot r')x$, $r(x + y) = rx + ry$, $(r + r')x = rx + r'x$,
2) axioms of lattices: $x \sqcup (y \sqcup z) = (x \sqcup y) \sqcup z$, $x \sqcup y = y \sqcup x$, $x \sqcup (x \sqcap y) = x$, $x \sqcap (y \sqcap z) = (x \sqcap y) \sqcup z$, $x \sqcap y = y \sqcap x$, $x \sqcap (x \sqcup y) = x$,
3) compatibility axioms: $(x \sqcap y) + z \leq y + z$ and $0 \leq r(x \sqcup 0)$, for all $r \in \mathbb{R}_{\geq 0}$

where $-x$ and $x \leq y$ are abbreviations for $(-1)x$ and $x \sqcap y = x$, respectively.

Hence the family of Riesz spaces is a variety in the sense of universal algebra.

Example II.10. The most familiar example is the real line $\mathbb{R}$ with its usual linear ordering, *i.e.*, with $\sqcup$ and $\sqcap$ being the usual $\max$ and $\min$ operations. An important fact about this Riesz space is the following (see, e.g., [24]). Given two terms t_1, t_2 in the language of Riesz spaces, the equality $t_1 = t_2$ holds in all Riesz spaces if and only if $t_1 = t_2$ is true in $\mathbb{R}$. In the terminology of universal algebra one says that $\mathbb{R}$ generates the variety of all Riesz spaces. In this sense $\mathbb{R}$ plays in the theory of Riesz spaces a role similar to the two-element Boolean algebra $\{0, 1\}$ in the theory of Boolean algebras.

Example II.11. For an example of Riesz space whose order is not linear take the vector space $\mathbb{R}^n$ with order defined pointwise: $(x_1, \ldots, x_n) \leq (y_1, \ldots, y_n) \Leftrightarrow x_i \leq y_i$, for all $1 \leq i \leq n$. More generally, for every set X, the set $\mathbb{R}^X = \{f : X \to \mathbb{R}\}$ with operations defined pointwise is a Riesz space. Since Riesz spaces are algebras, other examples can be found by taking sub-algebras. For instance, the collection of bounded functions $\{f \in \mathbb{R}^X \mid f \text{ is bounded}\}$ is a Riesz subspace of $\mathbb{R}^X$. As another example, if X is a topological space, then the set of continuous functions $C(X) = \{f \in \mathbb{R}^X \mid f \text{ is continuous}\}$ is another Riesz subspace of $\mathbb{R}^X$.

The following definitions are useful. Let A be a Riesz space. An element a is *positive* if $a \geq 0$. The set of all positive elements is called the *positive cone* and is denoted by A^+. Given an element $a \in A$, we define $a^+ = a \sqcup 0$, $a^- = -a \sqcup 0$ and $|a| = a^+ + a^-$. Note that $a^+, a^-, |a| \in A^+$, $a^+ = (-a)^-$, $a^- = (-a)^+$ and $a = a^+ - a^-$.

Definition II.12 (Archimedean Riesz space)**.** An element $a \neq 0$ of a Riesz space A is *infinitely small* if there exists some $b \in A$ such that $n|a| \leq |b|$, for all $n \in \mathbb{N}$. A Riesz space is *Archimedean* if it does not have any infinitely small element.

All the Riesz spaces in Example II.10 are Archimedean.

Example II.13. The vector space $\mathbb{R}^2$ with the lexicographic order, defined as $(x_1, y_1) \leq (x_2, y_2) \Leftrightarrow$ either $x_1 < x_2$ or $x_1 = x_2$ and $y_1 \leq y_2$, is not Archimedean. For instance, $(0, 1)$ is infinitely small with respect to $(1, 0)$.

As usual in universal algebra, a homomorphism between Riesz spaces is a function $f : A \to B$ preserving all operations. Therefore a *Riesz homomorphism* is a linear map preserving finite meets and joins.

Definition II.14 (Ideals and Maximal Ideals)**.** A subset $J \subseteq A$ of a Riesz space A is an *ideal* if it is the kernel of a homomorphism $f : A \to B$, *i.e.*, $J = f^{-1}(\{0\}) = \{a \mid f(a) = 0\}$, for some Riesz space B. The sets $\emptyset$ and A itself are trivially ideals. All other ideals are called *proper*. Ideals in A can be partially ordered by inclusion. An ideal $J \subseteq A$ is called *maximal* if it is a proper ideal and there is no larger proper ideal $J \subsetneq J'$.

We now introduce the important concept of a strong unit.

Definition II.15 (Strong Unit)**.** An element $u \in A$ is called a *strong unit* if it is positive (*i.e.*, $u \in A^+$) and for every $a \in A$ there exists $n \in \mathbb{N}$ such that $|a| \leq n(u)$.

Example II.16. The real line $\mathbb{R}$ has 1 as strong unit. The space $\mathbb{R}^{\mathbb{N}}$ does not have a strong unit. Its subspace consisting of bounded functions has $\mathbb{1}_{\mathbb{N}}$ (the constant $n \mapsto 1$ function) as strong unit. Similarly, let X be a compact topological space and $C(X)$ the Riesz space of continuous functions into $\mathbb{R}$. Since X is compact, any function $f \in C(X)$ is bounded and therefore $\mathbb{1}_X$ is a strong unit of $C(X)$.

We now introduce a notion of convergence in Riesz spaces which plays a role in the duality theory of Riesz spaces.

Definition II.17 (u-convergence and u-uniform Cauchy sequences)**.** Let A be a Riesz space and u be a positive element $u \geq 0$. We say that a sequence $(a_n)_{n\in\mathbb{N}}$ *converges u-uniformly* to b, written $(a_n) \to_u b$, if for every positive real $\epsilon > 0$

there exists a natural number N_ϵ such that $|b - a_n| \leq \epsilon u$, for all $n > N_\epsilon$. We say that $(a_n)_{n\in\mathbb{N}}$ is a *u-uniform Cauchy sequence* if for every $\epsilon > 0$ there exists a number N_ϵ such that $|a_i - a_j| \leq \epsilon u$, for all $i, j > N_\epsilon$.

Clearly, if $(a_n) \to_u b$ then (a_n) is a u-uniform Cauchy sequence.

Definition II.18 (uniform completeness)**.** A Riesz space A is *u-uniformly complete* if for every u-uniform Cauchy sequence (a_n) there exists $b \in A$ such that $(a_n) \to_u b$. It is *uniformly complete* if it is u-uniformly complete, for all $u \in A^+$.

We now state important properties related to uniform completeness of Archimedean Riesz spaces with strong unit.

Theorem II.19 (45.5 in [12])**.** *If A is Archimedean and has strong unit u, then A is uniformly complete if and only if it is u-uniform complete.*

Example II.20. The Riesz space $\mathbb{R}$ is 1-uniformly complete as the notion of 1-uniform Cauchy sequence coincides with the usual notion of Cauchy sequence of reals. Since 1 is a strong unit in $\mathbb{R}$ it follows that $\mathbb{R}$ is uniformly complete. Let X be a compact Hausdorff space, $C(X)$ the Riesz space of continuous functions $f: X \to \mathbb{R}$ and $\mathbb{1}_X \in C(X)$ the constant function $x \mapsto 1$. Then $C(X)$ is $\mathbb{1}_X$-uniformly complete ([12, Example 27.7, Theorem 43.1]). Once again, $C(X)$ is uniformly complete because $\mathbb{1}_X$ is a strong unit.

Theorem II.21 (Theorem 43.1 in [12])**.** *Let A be Archimedean with strong unit $u \in A$. Let $\|\_\| : A \to \mathbb{R}_{\geq 0}$ be defined as:*

$$\|a\| = \sup\{r \in \mathbb{R} \mid |a| \leq ru\} \tag{4}$$

Then $\|\_\|$ is a norm on A, i.e., $\|0\| = 0$, $\|a + b\| \leq \|a\| + \|b\|$ and $\|ra\| = |r| \cdot \|a\|$, for all $a, b \in A$ and $r \in \mathbb{R}$.

As a consequence, each Archimedean Riesz space with strong unit is a normed vector space and therefore can be endowed with the metric $d_A : A^2 \to \mathbb{R}_{\geq 0}$ defined as $d_A(a, b) = \|a - b\|$. Accordingly, we say that a Riesz homomorphism $f : A \to B$ between Archimedean spaces with strong units is *continuous* (resp. is an *isometry*) if it is continuous (resp. distance preserving) with respect to the metrics of A and B.

Importantly, on Archimedean spaces with strong unit, the notion of uniform convergence and convergence in the norm (i.e., in the metric d) coincide.

Theorem II.22 (Theorem 43.1 in [12])**.** *Let A be an Archimedean Riesz space with strong unit u. A sequence (a_n) converges u-uniformly to b if and only if (a_n) converges in norm to b. The space A is uniformly complete if and only if it is complete as a metric space.*

D. Categories of Riesz Spaces with a distinguished positive element

It is now convenient to extend the language of Riesz spaces with a new constant symbol u for a positive element.

Definition II.23. A Riesz space with *distinguished positive element* u is a pair (A, u) where A is a Riesz space and $u \geq 0$. A morphism between (A, u) and (B, v) is a Riesz homomorphism $f : A \to B$ such that $f(u) = v$. If u is a strong unit in A we say that (A, u) is *unital*.

When confusion might arise, we will stress the fact that a homomorphism $f : (A, u) \to (B, v)$ preserves the distinguished positive elements (*i.e.*, $f(u) = v$) by saying that f is a *unital (Riesz) homomorphism*. We write $\mathbf{Riesz}^u$ for the category having Riesz spaces (A, u) with a distinguished positive element as objects and unital homomorphisms as morphisms. We write $\mathbf{URiesz}$ for the subcategory of $\mathbf{Riesz}^u$ whose objects are unital Riesz spaces.

Example II.24. The basic example is the real line $(\mathbb{R}, 1)$. Since 1 is a strong unit, this is in fact a unital Riesz space. Furthermore it follows easily from the result mentioned in Example II.10 that $(\mathbb{R}, 1)$ generates the variety $\mathbf{Riesz}^u$.

The following theorem (see, e.g., [12, Thm 27.3-4]) expresses a key property of unital Riesz spaces.

Theorem II.25. *Let (A, u) be a unital Riesz space. Then, for every unital homomorphism $f : (A, u) \to (\mathbb{R}, 1)$, the ideal $f^{-1}(0)$ is maximal. Conversely, every maximal ideal J in (A, u) is of the form $f_J^{-1}(0)$ for a unique unital Riesz homomorphism $f_J : (A, u) \to (\mathbb{R}, 1)$.*

Hence there is a one-to-one correspondence between maximal ideals in unital Riesz spaces (A, u) and homomorphisms into $(\mathbb{R}, 1)$ preserving the unit. Observe, once again (cf. Examples II.10 and II.24), how the Riesz space $(\mathbb{R}, 1)$ plays in the theory of unital Riesz spaces a role similar to two element Boolean algebra $\{0, 1\}$ in the theory of Boolean algebras.

The following theorem describes the property of being Archimedean for unital Riesz spaces.

Theorem II.26. *Let (A, u) be a unital Riesz space. Then A is Archimedean if and only if for every $a \neq 0$ there exists a unital Riesz homomorphism $f : (A, u) \to (\mathbb{R}, 1)$ such that $f(a) \neq 0$.*

We say that a unital Riesz space (A, u) is Archimedean if A is Archimedean. We write $\mathbf{AURiesz}$ for the category of Archimedean unital Riesz spaces with unital Riesz homomorphisms. We write $\mathbf{CAURiesz}$ for the category of Archimedean and uniformly complete unital Riesz spaces with unital Riesz homomorphisms.

$$\mathbf{CAURiesz} \hookrightarrow \mathbf{AURiesz} \hookrightarrow \mathbf{URiesz} \hookrightarrow \mathbf{Riesz}^u$$

Example II.27. Let X be a compact Hausdorff space. Let $\mathbb{1}_X$ be the constant $(x \mapsto 1) \in C(X)$ function. Then $(C(X), \mathbb{1}_X)$ is an Archimedean unital and uniformly complete Riesz space. [12, Example 27.7, Theorem 43.1]

E. Yosida's Theorem and Duality Theory of Riesz Spaces

In this section we assume familiarity with the basic notions from category theory regarding equivalences of categories and adjunctions. A standard reference is [25].

The celebrated Stone duality theorem states that any Boolean algebra B is isomorphic to the Boolean algebra of clopen sets (or equivalently continuous functions $f : X \to$

$\{0,1\}$ where $\{0,1\}$ is given the discrete topology) of a unique (up to homeomorphism) *Stone space*, *i.e.*, a compact Hausdorff and zero–dimensional topological space X. Here X is the collection $\mathrm{Spec}(B)$ of maximal (Boolean) ideals in B endowed with the *hull–kernel* topology. In fact this correspondence can be made into a *categorical equivalence* between $\mathbf{Stone}$ and $\mathbf{Bool}^{\mathrm{op}}$.

A similar representation theorem, due to Yosida [26], states that every uniformly complete, unitary and Archimedean Riesz space (A,u) is isomorphic to $(C(X),\mathbb{1}_X)$, the Riesz space of all continuous functions $f:X\to\mathbb{R}$, of a unique (up to homeomorphism) compact Hausdorff space X. This correspondence can be made into a categorical equivalence

$$\mathbf{CHaus} \simeq \mathbf{CAURiesz}^{\mathrm{op}} \tag{5}$$

see, e.g., [27] for a detailed proof. In fact Yosida proved a more general result which can be conveniently formulated as an adjunction between $\mathbf{CHaus}$ and $\mathbf{AURiesz}^{\mathrm{op}}$ which restricts to the equivalence (5) on the subcategory $\mathbf{CAURiesz}^{\mathrm{op}}$. In the rest of this section we describe it as a unit–counit adjunction $(\eta,\epsilon): C \dashv \mathrm{Spec}$ consisting of two functors:

$$C:\mathbf{CHaus}\to\mathbf{CAURiesz}^{\mathrm{op}}\hookrightarrow\mathbf{AURiesz}^{\mathrm{op}}$$
$$\mathrm{Spec}:\mathbf{AURiesz}^{\mathrm{op}}\to\mathbf{CHaus}$$

and two natural transformations:

$$\eta:\mathrm{id}_{\mathbf{CHaus}}\to\mathrm{Spec}\circ C$$
$$\epsilon: C\circ\mathrm{Spec}\to\mathrm{id}_{\mathbf{AURiesz}^{\mathrm{op}}}$$

called unit and counit, respectively.

We first define the functor $C:\mathbf{CHaus}\to\mathbf{CAURiesz}^{\mathrm{op}}$.

On objects, for a compact Hausdorff space X, we define $C(X)$ as the set of continuous real-valued functions on X, equipped with the Riesz space operations defined pointwise from those on $\mathbb{R}$ (see Example II.11) and strong unit $\mathbb{1}_X$ ($x\mapsto 1$). As discussed earlier (see Example II.27) this is indeed a uniformly complete Archimedean and unital Riesz space. On continuous maps $f:X\to Y$, we define $C(f)(b)=b\circ f$, for all $b\in C(Y)$. This is easily proven to be a unital Riesz space morphism by the fact that the Riesz space operations are defined pointwise.

We now turn our attention to the description of the functor $\mathrm{Spec}:\mathbf{AURiesz}^{\mathrm{op}}\to\mathbf{CHaus}$.

As in the Stone duality theorem, on objects (A,u) in $\mathbf{AURiesz}$, the functor $\mathrm{Spec}(A)$ is defined as the *spectrum of* A, *i.e.*, the collection of all maximal ideals of A (see Definition II.14) equipped with the hull–kernel topology which can be defined as follows. A subset $X\subseteq\mathrm{Spec}(A)$ is closed in the hull–kernel topology if and only if there exists a (not necessarily maximal) ideal $I\subseteq A$ such that $X=\mathrm{hull}(I)$ where $\mathrm{hull}(I)=\{J\in\mathrm{Spec}(A)\mid I\subseteq J\}$. See, e.g., [12, Theorem 36.4 (ii)] for a proof that $\mathrm{Spec}(A)$ is indeed a compact Hausdorff space. On maps, for a unital morphism $f:(A,u_A)\to(B,u_B)$ we define, for every $J\in\mathrm{Spec}(B)$, $\mathrm{Spec}(f)(J)=f^{-1}(J)$.

We now turn our attention to the description of the unit map $\eta:\mathrm{id}_{\mathbf{CHaus}}\to\mathrm{Spec}\circ C$.

This is a collection of maps $\{\eta_X : X\to\mathrm{Spec}(C(X))\}$ indexed by compact Hausdorff spaces. For a fixed compact Hausdorff space X and $x\in X$ we can define the map $\delta_x: C(X)\to\mathbb{R}$ as $\delta_x(f)=f(x)$ which is easily seen to be a unital Riesz homomorphism. Therefore, by Theorem II.25 the set $N_x=\delta_x^{-1}(0)$ is a maximal ideal in $C(X)$, i.e., $N_x\in\mathrm{Spec}(C(X))$. We then define η_X as $\eta_X(x)=N_x$.

Lastly, we now proceed with the definition of the counit map $\epsilon: C\circ\mathrm{Spec}\to\mathrm{id}_{\mathbf{AURiesz}^{\mathrm{op}}}$.

This is a collection of morphisms $\{\epsilon_A : C(\mathrm{Spec}(A))\to A\}$ in $\mathbf{AURiesz}^{\mathrm{op}}$, or equivalently a collection of morphisms $\{\epsilon_A : A\to C(\mathrm{Spec}(A))\}$ in $\mathbf{AURiesz}$, indexed by unital and Archimedean Riesz spaces (A,u_A). For a fixed such (A,u) and $a\in A$ we can define a function $\hat{a}:\mathrm{Spec}(A)\to\mathbb{R}$ as $\hat{a}(J)=f_J(a)$, where f_J is the homomorphism from Theorem II.25. That is (see [12, Thm 27.3-4]) the value $\hat{a}(J)$ is defined as the unique real number r such that $ru_A-a\in J$. The map $\hat{a}$ is continuous, i.e., $\hat{a}\in C(\mathrm{Spec}(A))$. We then define ϵ_A as $\epsilon_A(a)=\hat{a}$.

The statement of Yosida's theorem can then be formulated by the following two theorems (see [26, Theorems 1–3], also [12, Theorems 45.3 and 45.4] and [27]).

Theorem II.28. *Both C and Spec are functors. Both η and ϵ are natural transformations. The quadruple $(\eta,\epsilon): C\dashv\mathrm{Spec}$ is a unit-counit adjunction. The counit map ϵ_A is an isometric isomorphism between A and its image in $C(\mathrm{Spec}(A))$.*

Theorem II.29. *When restricted to $\mathbf{CAURiesz}^{\mathrm{op}}$, the adjunction becomes an equivalence of categories. An object (A,u) of $\mathbf{AURiesz}$ is uniformly complete (i.e., it belongs to $\mathbf{CAURiesz}$) if and only if ϵ_A is a Riesz isomorphism.*

The functor $C\circ\mathrm{Spec}:\ \mathbf{AURiesz}^{\mathrm{op}}\to\mathbf{CAURiesz}^{\mathrm{op}}$ maps (not necessarily uniformly complete) Archimedean unital Riesz spaces to uniformly complete ones. In fact, Yosida showed that A embeds densely in $C(\mathrm{Spec}(A))$. Therefore $C(\mathrm{Spec}(A))$ is isomorphic to the completion of A in its norm (from Theorem II.21).

Definition II.30. The uniform Archimedean and unital Riesz space $C(\mathrm{Spec}(A))$ is called the *uniform completion of* A and is simply denoted by $\hat{A}$. We always identify A with the (isomorphic) dense sub-Riesz space $\epsilon_A(A)$ of $\hat{A}$.

Proposition II.31. *For every $A\in\mathbf{AURiesz}$, the two spaces $\mathrm{Spec}(A)$ and $\mathrm{Spec}(\hat{A})$ are homeomorphic. Furthermore, for every $B\in\mathbf{AURiesz}$ and unital homomorphism $f:A\to B$ there exists a unique unital Riesz homomorphism $\hat{f}:\hat{A}\to\hat{B}$ extending A.*

III. Riesz Modal Logic for Markov Processes

In this section we formally introduce the Riesz modal logic for Markov processes.

Definition III.1 (Syntax)**.** The set of formulas `Form` is generated by the following grammar:

$$\phi,\psi ::= 0 \mid 1 \mid r\phi \mid \phi{+}\psi \mid \phi{\sqcup}\psi \mid \phi{\sqcap}\psi \mid \Diamond\phi \qquad \text{where } r\in\mathbb{R}.$$

The semantics of a formula ϕ, interpreted on a Markov process $\alpha: X\to\mathcal{R}^{\leq 1}(X)$ (see Definition II.7) is a continuous function $[\![\phi]\!]_\alpha : X\to\mathbb{R}$ defined as follows.

Definition III.2 (Semantics)**.** Let $\alpha : X \to \mathcal{R}^{\leq 1}(X)$ be a Markov process. The semantics (or interpretation) of a formula ϕ by the Markov process α is the continuous function $[\![\phi]\!]_\alpha \in C(X)$ defined by induction on ϕ as follows:

$$[\![0]\!]_\alpha(x) = 0 \qquad [\![1]\!]_\alpha(x) = 1$$

$$[\![r\phi]\!]_\alpha(x) = r\cdot\big([\![\phi]\!]_\alpha(x)\big) \qquad [\![\phi+\psi]\!]_\alpha(x) = [\![\phi]\!]_\alpha(x)+[\![\psi]\!]_\alpha(x)$$

$$[\![\phi \sqcup \psi]\!]_\alpha(x) = \max\big\{[\![\phi]\!]_\alpha(x), [\![\psi]\!]_\alpha(x)\big\}$$

$$[\![\phi \sqcap \psi]\!]_\alpha(x) = \min\big\{[\![\phi]\!]_\alpha(x), [\![\psi]\!]_\alpha(x)\big\}$$

$$[\![\Diamond\phi]\!]_\alpha(x) = \int_X [\![\phi]\!]_\alpha \,\mathrm{d}\alpha(x) = \mathbb{E}_{\alpha(x)}([\![\phi]\!]_\alpha)$$

Hence $[\![0]\!]_\alpha$ and $[\![1]\!]_\alpha$ are the constants functions $\mathbb{0}_X$ $(x \mapsto 0)$ and $\mathbb{1}_X$ $(x \mapsto 1)$, respectively. The connectives $\{r(\_), +, \sqcup, \sqcap\}$ correspond to the real vector space and lattice operations of $\mathbb{R}$ lifted to $C(X)$ pointwise (see examples and II.11 and II.27). The semantics of the formula $\Diamond\phi$ is the function that assignes to x the expected value of $[\![\phi]\!]_\alpha$ with respect to the subprobability measure $\alpha(x)$. The fact that $[\![\Diamond\phi]\!]_\alpha$ is indeed continuous (see Lemma III.5 below) is a direct consequence of the Riesz–Markov–Kakutani as we now explain.

Recall from Section II-A that, by the Riesz–Markov–Kakutani theorem, we have the correspondence

$$\mathcal{R}^{\leq 1}(X) \simeq (X \xrightarrow{c} \mathbb{R}) \xrightarrow{l} \mathbb{R} \qquad\qquad \mu \longleftrightarrow \mathbb{E}_\mu$$

where we used the letters c and l as a reminder of when the space of continuous functions and the space of positive, linear and $\mathbb{1}_X$-decreasing functions are considered. Therefore, each Markov process $\alpha : X \to \mathcal{R}^{\leq 1}(X)$ can be identified as the function:

$$\alpha : X \xrightarrow{c} \big((X \xrightarrow{c} \mathbb{R}) \xrightarrow{l} \mathbb{R}\big)$$

where

$$\alpha(x)(f) = \mathbb{E}_{\alpha(x)}(f) = \int_X f \,\mathrm{d}\alpha(x).$$

By swapping the arguments of α as a curried function, we obtain a positive linear map $C(X) \to C(X)$ (where $C(X) = X \xrightarrow{c} \mathbb{R}$) which, for clarity, we denote by $\Diamond_\alpha$:

$$\Diamond_\alpha : (X \xrightarrow{c} \mathbb{R}) \to (X \xrightarrow{c} \mathbb{R}), \text{ where } \Diamond_\alpha(f)(x) = \alpha(x)(f) \quad (6)$$

To see that $\Diamond_\alpha(f)$ is indeed a continuous function, for any $f \in C(X)$, let $(x_i)_{i\in I}$ be a net in X converging to $x \in X$. We need to prove that $\lim_{i\in I} \Diamond_\alpha(f)(x_i) = \Diamond_\alpha(f)\,(\lim_{i\in I} x_i)$. This follows from the definition (6) and from

$$\lim_{i\in I} \alpha(x_i)(f) = \left(\lim_{i\in I} \alpha(x_i)\right)(f) = \alpha\left(\lim_{i\in I} x_i\right)(f)$$

where the first equality follows from the definition of the weak-* topology and the second from the continuity of α.

The following proposition then follows.

Proposition III.3. *Let $\alpha : X \to \mathcal{R}^{\leq 1}(X)$ be a Markov process and let $\Diamond_\alpha$ be defined as above. Then, for every $f, g \in C(X)$, the operator $\Diamond_\alpha$ has the following properties:*

- *(Linear)* $\Diamond_\alpha(r_1 f + r_2 g) = r_1\Diamond_\alpha(f) + r_2\Diamond_\alpha(g)$
- *(Positive) if* $f \geq \mathbb{0}_X$ *then* $\Diamond_\alpha(f) \geq \mathbb{0}_X$,
- *($\mathbb{1}_X$-decreasing)* $\Diamond_\alpha(\mathbb{1}_X) \leq \mathbb{1}_X$.

This discussion allows us to equivalently rephrase the definition of the semantics of Riesz modal logic formulas.

Definition III.4 (Semantics, rephrased)**.** Let $\alpha : X \to \mathcal{R}^{\leq 1}(X)$ be a Markov process. The semantics $[\![\phi]\!]_\alpha \in C(X)$ of ϕ can be defined by induction on ϕ as follows:

$$[\![0]\!]_\alpha = \mathbb{0}_X \qquad [\![1]\!]_\alpha = \mathbb{1}_X$$

$$[\![r\phi]\!]_\alpha = r[\![\phi]\!]_\alpha \quad [\![\phi+\psi]\!]_\alpha = [\![\phi]\!]_\alpha + [\![\psi]\!]_\alpha$$

$$[\![\phi \sqcup \psi]\!]_\alpha = [\![\phi]\!]_\alpha \sqcup [\![\psi]\!]_\alpha \qquad [\![\phi \sqcap \psi]\!]_\alpha = [\![\phi]\!]_\alpha \sqcap [\![\psi]\!]_\alpha$$

$$[\![\Diamond\phi]\!]_\alpha = \Diamond_\alpha([\![\psi]\!]_\alpha)$$

The following lemma now becames obvious since $\Diamond_\alpha$ maps continuous functions to continuous functions.

Lemma III.5. *For every ϕ the function $[\![\phi]\!]_\alpha$ is continuous.*

The following proposition states that the the semantics of formulas is invariant under coalgebra morphisms.

Proposition III.6. *Let $\alpha : X \to \mathcal{R}^{\leq 1}(X)$ and $\beta : Y \to \mathcal{R}^{\leq 1}(Y)$ be two Markov processes and let $\alpha \xrightarrow{f} \beta$ be a coalgebra morphism. For every formula ϕ the equality $[\![\phi]\!]_\alpha = [\![\phi]\!]_\beta \circ f$ holds, i.e., $[\![\phi]\!]_\alpha(x) = [\![\phi]\!]_\beta(f(x))$, for all $x \in X$.*

Proof. Simple unfolding of definitions (see Appendix). □

We now turn our attention to the set of valid equalities between modal Riesz formulas.

Definition III.7 (Equivalence of formulas)**.** Given a Markov process $\alpha : X \to \mathcal{R}^{\leq 1}(X)$, we say that two formulas ϕ and ψ are *α-equivalent*, written $\phi \sim_\alpha \psi$, if it holds that $[\![\phi]\!]_\alpha = [\![\psi]\!]_\alpha$. Similarly, we say that two formulas are equivalent, written $\phi \sim \psi$, if for all $\alpha \in \mathbf{Markov}$ it holds that $\phi \sim_\alpha \psi$.

It is clear, from the unital Riesz space structure of $(C(X), \mathbb{1}_X)$, that all Riesz spaces axioms hold true with respect to the equivalence relation $\sim$. For example $\phi+\psi \sim \psi+\phi$ and $(r+s)\phi \sim r\phi + s\psi$. It also follows from the previous discussion on the semantics of the formula $\Diamond\phi$ that

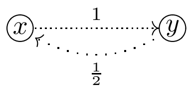
- (Linearity) $r\Diamond\phi \sim \Diamond(r\phi)$ and $\Diamond(\phi+\psi) \sim \Diamond\phi + \Diamond\psi$
- (Positivity) $\Diamond(\phi \sqcup 0) \sqcup 0 \sim \Diamond(\phi \sqcup 0)$
- ($\mathbb{1}$-decreasing) $\Diamond(1) \sqcup 1 \sim 1$

It will be one of the main goals of this work to show that, in fact, this set of axioms (axioms of Riesz spaces with a positive element together with the axioms listed above for $\Diamond$) is *complete* in the sense that any valid equality $\phi \sim \psi$ can be derived syntactically from these axioms using the inference rules of equational logic. This is stated precisely as Theorem VII.1 in Section VII.

A. Examples of formulas and their meaning

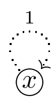
In this subsection we consider some some examples of Riesz modal formulas to familiarize with the previous definitions.

Example III.8. Let us consider the Markov processes $\alpha : X \to \mathcal{R}^{\leq 1}(X)$ with finite state space $X = \{x, y\}$ defined by: $\alpha(x) = \{0 : x, 1 : y\}$ and $\alpha(y) = \{\frac{1}{2} : x, 0 : y\}$,



and consider the Riesz modal logic formula $\Diamond 1$. The formula $\Diamond 1$ can be understood as mapping each state $x \in X$ to the total mass of the subprobability measure $\alpha(x)$. In the above depicted example we have $[\![\Diamond 1]\!]_\alpha(x) = 1$ and $[\![\Diamond 1]\!]_\alpha(y) = \frac{1}{2}$.

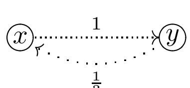
Example III.9. Let us consider the Markov processes $\beta : X \to \mathcal{R}^{\leq 1}(X)$ with state space $X = \{x\}$ defined by: $\beta(x) = \{1 : x\}$,



We then have that $[\![\Diamond 1]\!]_\beta(x) = 1$. Note, therefore, that $1 \sim_\beta \Diamond 1$ holds, i.e., the formula 1 and $\Diamond 1$ are β-equivalent. However $1 \not\sim \Diamond 1$ as the previous example shows.

Example III.10. Consider the formula $-\Diamond 1 + 1$. In the Markov process α of Example III.8 we have $[\![-\Diamond 1 + 1]\!]_\alpha(x) = 0$ and $[\![-\Diamond 1 + 1]\!]_\alpha(y) = \frac{1}{2}$. Generally, one can observe that for every Markov process γ, $[\![-\Diamond 1 + 1]\!]_\gamma(x) \in [0, 1]$ with $[\![-\Diamond 1 + 1]\!]_\gamma(x)$ assuming values 0 and 1 if and only if $\gamma(x)$ has total mass 1 and 0, respectively.

Example III.11. Consider now the formula $\phi = \Diamond 1 \sqcup (-\Diamond 1 + 1)$. This assigns to a state x the maximum between the valued assigned to x by $\Diamond 1$ and $(-\Diamond 1 + 1)$. In the Markov process α of Example III.8 we have $[\![\phi]\!]_\alpha(x) = 1$ and $[\![\phi]\!]_\alpha(y) = \frac{1}{2}$. Hence we have $\Diamond 1 \sim_\alpha \phi$. However this is not a valid equality in general. A counterexample is given by a variant of α defined as $\alpha'(x) = \{0 : x, 1 : y\}$ and $\alpha'(y) = \{\frac{1}{3} : x, 0 : y\}$ as depicted below:



Indeed $[\![\Diamond 1]\!]_{\alpha'}(y) = \frac{1}{3}$ and $[\![\phi]\!]_{\alpha'}(y) = \frac{2}{3}$.

Formulas themselves represent real valued continuous functions. But equalities between formulas can capture interesting classes of Markov processes.

Example III.12. Consider the equality $\Diamond\Diamond 1 = 0$. Clearly this is not a valid equality. For example in the Markov process α of Example III.8 we have $\Diamond\Diamond 1 \not\sim_\alpha 1$. The equality holds precisely on the class of Markov processes $\gamma : X \to \mathcal{R}^{\leq 1}(X)$ where for each $x \in X$ either $\alpha(x)$ has mass 0 (i.e., $[\![\Diamond 1]\!]_\gamma(x) = 0$) or $\alpha(x)$ assigns probability 1 to the set of states y such that $\alpha(y)$ has mass 0. In other words, the equality $\Diamond\Diamond 1 = 0$ holds in those systems where from each state x the probability of making two consecutive steps is 0.

Example III.13. Consider the equality $\Diamond 1 = (\Diamond 1 + \Diamond 1) \sqcap 1$. This equality holds precisely on the class of Markov processes $\gamma : X \to \mathcal{R}^{\leq 1}(X)$ such that for every $x \in X$ either $\alpha(x)$ is a probability measure (i.e., having mass 1) or the null measure (i.e., having mass 0). Indeed if $\alpha(x)$ has mass $0 \lneq m \lneq 1$ then $[\![\Diamond 1]\!]_\gamma(x) = m$ and $[\![(\Diamond 1 + \Diamond 1) \sqcap 1]\!]_\gamma(x) = \min\{m + m, 1\} \gneq m$. For the reader familiar with coalgebra, this class of Markov processes can be identified as the F-coalgebras in the category **CHaus** for the functor $F(X) = 1 + \mathcal{R}^{=1}(X)$.

B. Relation with other logics in the literature

Other real-valued logics for expressing properties of Markov processes or similar systems (e.g., Markov decision processes, weighted systems, etc.) differ from the Riesz modal logic in the choice of the basic connectives. It turns out that most of such logics can be interpreted within the Riesz modal logic.

For example, the modal logic of Panangaden (see [11, §8.2]), which is particularly important because it characterizes the Kantorovich pseudo-metric on Markov processes, has real-valued semantics of type $[\![\phi]\!]_\alpha : X \to [0, 1]$ with formulas defined by the syntax: $\phi, \psi ::= 1 \mid 1 - \phi \mid \phi \sqcap \psi \mid \Diamond\phi \mid \phi \ominus r$, where $r \in [0, 1]$ and $[\![\phi \ominus r]\!]_\alpha(x) = \max\{0, [\![\phi]\!]_\alpha(x) - r\}$. Therefore this logic can be directly interpreted in the Riesz modal logic by defining $\phi \ominus r = 0 \sqcup (\phi - r1)$.

Similarly, the modal logic underlying the Łukasiewicz modal μ-calculus (see [8] and [9]), which is important because this logic is sufficiently expressive to interpret pCTL, has also real-valued semantics of type $[\![\phi]\!]_\alpha : X \to [0, 1]$ with formulas defined by the syntax:

$$\phi, \psi ::= 0 \mid 1 \mid r\phi \mid \phi \oplus \psi \mid \phi \odot \psi \mid \phi \sqcup \psi \mid \phi \sqcap \psi \mid \Diamond\phi$$

where $r \in [0, 1]$ and $[\![\phi \oplus \psi]\!]_\alpha(x) = \min\{1, [\![\phi]\!]_\alpha(x) + [\![\phi]\!]_\alpha(x)\}$ and $[\![\phi \odot \psi]\!]_\alpha(x) = \max\{0, [\![\phi]\!]_\alpha(x) + [\![\phi]\!]_\alpha(x) - 1\}$. Therefore, also this logic can be interpreted in the Riesz modal logic by defining $\phi \oplus \psi = 1 \sqcap (\phi + \psi)$ and $\phi \odot \psi = 0 \sqcup (\phi + \psi - 1)$.

IV. Modal Riesz Spaces

In this section we introduce the notion of modal Riesz space. This will be the variety of algebras corresponding to the Riesz modal logic for Markov processes.

Definition IV.1. A *modal Riesz space* is a structure $(A, u, \Diamond)$ where (A, u) is a Riesz space with designated positive element u (Definition II.23) and $\Diamond : A \to A$ is a unary function such that:

1) (Linearity) $\Diamond(a + b) = \Diamond(a) + \Diamond(b)$ and $\Diamond(ra) = r(\Diamond a)$, for all $r \in \mathbb{R}$
2) (Positivity) $\Diamond(a \sqcup 0) \geq 0$,
3) (u-decreasing) $\Diamond(u) \leq u$.

Thus the class of modal Riesz spaces is a variety in the sense of universal algebra. Homomorphisms of modal Riesz spaces are unital Riesz homomorphisms which further preserve the $\Diamond$ function (i.e., $f(\Diamond(a)) = \Diamond(f(a))$. We say that $(A, u, \Diamond)$ is *Archimedean* (resp. *unital* and *u-complete*) if (A, u) is

Archimedean (resp. unital and u-complete). We denote by $\mathbf{Riesz}^u_\Diamond$ the category having modal Riesz spaces as objects and homomorphisms of modal Riesz spaces as morphisms. We also define $\mathbf{URiesz}_\Diamond$, $\mathbf{AURiesz}_\Diamond$ and $\mathbf{CAURiesz}_\Diamond$ to be the categories of unital, Archimedean and unital, u-complete Archimedean and unital modal Riesz spaces, respectively.

$$\mathbf{CAURiesz}_\Diamond \hookrightarrow \mathbf{AURiesz}_\Diamond \hookrightarrow \mathbf{URiesz}_\Diamond \hookrightarrow \mathbf{Riesz}^u_\Diamond$$

Remark IV.2. Note that in the presence of linearity, positivity of $\Diamond$ is equivalent to monotonicity of $\Diamond$ (i.e., $a \leq b$ implies $\Diamond(a) \leq \Diamond(b)$). Clearly monotonicity implies positivity. In the other direction, assume $\Diamond$ is positive and let $a \leq b$. Note that $a \leq b \Leftrightarrow b - a \geq 0$ [12, Thm 11.4]. Then by positivity $\Diamond(b-a) \geq 0$. By linearity, $\Diamond(b) - \Diamond(a) \geq 0$ and this is equivalent to $\Diamond(b) \geq \Diamond(a)$.

Example IV.3. As a trivial example, note that every Riesz space (A, u) can be given the structure of a modal Riesz space by taking, e.g., $\Diamond$ to be the constant 0 function $\Diamond(a) = 0$ or the identity function $\Diamond(a) = a$.

More interestingly, each Markov process gives rise to a modal Riesz space.

Example IV.4. Let $\alpha : X \to \mathcal{R}^{\leq 1}(X)$ be a Markov process. As discussed in Section III we can view α as the operator $\Diamond_\alpha : C(X) \to C(X)$ acting on the Riesz space $(C(X), \mathbb{1}_X)$. By Proposition III.3 the operator $\Diamond_\alpha$ satisfies the required properties to make $(C(X), \mathbb{1}_X, \Diamond_\alpha)$ a modal Riesz space. Furthermore, since $(C(X), \mathbb{1}_X) \in \mathbf{CAURiesz}$ we have that $(C(X), \mathbb{1}_X, \Diamond_\alpha) \in \mathbf{CAURiesz}_\Diamond$.

Hence, to each Markov process $\alpha : X \to \mathcal{R}^{\leq 1}(X)$ corresponds the modal Riesz space $A_\alpha = (C(X), \mathbb{1}_X, \Diamond_\alpha) \in \mathbf{CAURiesz}_\Diamond$.

By combining the Riesz–Markov–Kakutani representation theorem and Yosida's theorem we have in fact that this correspondence is bijective on isomorphism classes.

Theorem IV.5. *For each $A = (A, u, \Diamond) \in \mathbf{CAURiesz}_\Diamond$, given a choice of isomorphism $A \cong C(X)$, there exists one and only one Markov process $\alpha \in \mathbf{Markov}$ such that $A \cong A_\alpha$.*

Proof. By Yosida's theorem (Theorem II.29), (A, u) is isomorphic to $(C(X), \mathbb{1}_X)$ for a unique (up to homeomorphism) compact Hausdorff space $X = \mathrm{Spec}(A)$. Fixing such an isomorphism and conjugating the original $\Diamond$ by the isomorphism, we get a positive linear $\mathbb{1}_X$-decreasing map $\Diamond : C(X) \xrightarrow{l} C(X)$:

$$\Diamond : (X \xrightarrow{c} \mathbb{R}) \xrightarrow{l} (X \xrightarrow{c} \mathbb{R})$$

and by swapping the arguments as a curried function, we equivalently get a function which, for clarity, we denote by $\alpha_\Diamond$:

$$\alpha_\Diamond : X \xrightarrow{c} \big((X \xrightarrow{c} \mathbb{R}) \xrightarrow{l} \mathbb{R}\big) \qquad \alpha_\Diamond(x)(f) = \Diamond(f)(x),$$

By using the Riesz–Markov–Kakutani theorem, the space $\big((X \xrightarrow{c} \mathbb{R}) \xrightarrow{l} \mathbb{R}\big)$ coincides with $\mathcal{R}^{\leq 1}(X)$. We can show that $\alpha_\Diamond$ is indeed continuous using the definition of continuity in terms of nets, as follows. Let $(x_i)_{i \in I}$ be a net converging to $x \in X$. Since $\Diamond(f)$ is a continuous function, for each $f \in C(X)$, we have $\Diamond(f)(\lim_i x_i) = \lim_i(\Diamond(f)(x_i))$ and therefore, from the definition $\alpha_\Diamond$ we have

$$\alpha_\Diamond(\lim_i x_i)(f) = \lim_i(\alpha_\Diamond(x_i)(f))$$

As this holds for all $f \in C(X)$, this shows that $\alpha_\Diamond(\lim_i x_i) = \lim_i \alpha_\Diamond(x_i)$, where the latter limit is with respect to the weak-* topology, and proves that $\alpha_\Diamond$ is continuous.

Therefore we can see that $\alpha_\Diamond : X \to \mathcal{R}^{\leq 1}(X)$ is the unique Markov process corresponding to $(A, u, \Diamond)$. $\square$

Example IV.6. For a fixed compact Hausdorff space X, let $\alpha : X \to \mathcal{R}^{\leq 1}(X)$ be the Markov process defined as $\alpha(x) = \delta_x$, for all $x \in X$, where $\delta_x \in \mathcal{R}^{\leq 1}(X)$ is the *Dirac measure* defined as $\delta_x(A) = 1$ if $x \in A$ and $\delta_x(A) = 0$ otherwise, for all Borel sets $A \subseteq X$. More colloquially, α is the Markov process where each state $x \in X$ loops back to itself with probability 1. Let $A_\alpha = (C(X), \mathbb{1}_X, \Diamond_\alpha)$ be the modal Riesz space corresponding to α. It is easy to check that $\Diamond_\alpha$ is just the identity map, i.e., $\Diamond_\alpha(f) = f$, for all $f \in C(X)$.

Hence there is a bijective correspondence between the (isomorphism classes of) objects of $\mathbf{Markov}$ and the objects of $\mathbf{CAURiesz}_\Diamond$. It will be shown in the next section that this correspondence lifts to a *duality* between the two categories.

Remark IV.7. We have seen that Markov processes can be identified with modal Riesz spaces in $\mathbf{CAURiesz}_\Diamond$. However the properties of being Archimedean, unital and complete are not equationally (nor elementary) definable in the language of modal Riesz spaces. The larger category $\mathbf{Riesz}^u_\Diamond$ contains many objects which do not have these nice properties. It is suggestive to consider these objects as *nonstandard* Markov processes. Hence, when studying Markov processes through the lens of algebra (the variety $\mathbf{Riesz}^u_\Diamond$) one has always to consider the existence of such nonstandard models.

V. Duality

In this section we extend the adjunction $(\eta, \epsilon) : C \dashv \mathrm{Spec}$ between $\mathbf{CHaus}$ and $\mathbf{AURiesz}^{\mathrm{op}}$ of Section II-E to one between $\mathbf{Markov}$ and $\mathbf{AURiesz}^{\mathrm{op}}_\Diamond$ which becomes a duality when restricted to the subcategory $\mathbf{CAURiesz}^{\mathrm{op}}_\Diamond$. The unit-counit adjunction is described by the quadruple $(\eta^\Diamond, \epsilon^\Diamond) : C^\Diamond \dashv \mathrm{Spec}^\Diamond$ consisting of the two functors:

$$C^\Diamond : \mathbf{Markov} \to \mathbf{CAURiesz}^{\mathrm{op}}_\Diamond \hookrightarrow \mathbf{AURiesz}^{\mathrm{op}}_\Diamond$$
$$\mathrm{Spec}^\Diamond : \mathbf{AURiesz}^{\mathrm{op}}_\Diamond \to \mathbf{Markov}$$

and the two natural transformations:

$$\eta^\Diamond : \mathrm{id}_{\mathbf{Markov}} \to \mathrm{Spec}^\Diamond \circ C^\Diamond$$
$$\epsilon^\Diamond : C^\Diamond \circ \mathrm{Spec}^\Diamond \to \mathrm{id}_{\mathbf{AURiesz}^{\mathrm{op}}_\Diamond}$$

We start by defining the functor $C^\Diamond : \mathbf{Markov} \to \mathbf{CAURiesz}_\Diamond$. On objects $\alpha : X \to \mathcal{R}^{\leq 1}(X)$ in $\mathbf{Markov}$, it is defined as $C^\Diamond(\alpha) = A_\alpha = (C(X), \mathbb{1}_X, \Diamond_\alpha)$, as in (6) and Proposition III.3. On (coalgebra) maps $\alpha \xrightarrow{f} \beta$ between $\alpha : X \to \mathcal{R}^{\leq 1}(X)$ and $\beta : Y \to \mathcal{R}^{\leq 1}(Y)$ having underlying function $f : X \to Y$, we define $C^\Diamond(f)$ to be $C(f)$, where

C : $\mathbf{CHaus} \to \mathbf{CAURiesz}^{\mathrm{op}}$ is the functor described in Section II-E.

We now turn our attention to the definition of the functor $\mathrm{Spec}^{\Diamond} : \mathbf{AURiesz}_{\Diamond}^{\mathrm{op}} \to \mathbf{Markov}$. On objects $A = (A, u, \Diamond)$ belonging to $\mathbf{CAURiesz}_{\Diamond}$ the Markov process

$$\alpha_{\Diamond} : \mathrm{Spec}(A) \to \mathcal{R}^{\leq 1}(\mathrm{Spec}(A))$$

is defined as in Theorem IV.5. If instead A just belongs to $\mathbf{AURiesz}_{\Diamond}$ we only have (Theorem II.29) that A is isomorphic, via the counit map $\epsilon_A(a) = \hat{a}$, to a dense subspace of $C(\mathrm{Spec}(A))$. In this case, for each $J \in \mathrm{Spec}(A)$, we give a partial definition of the subprobability measure (seen as a linear functional) $\alpha_{\Diamond}(J)$ on all functions $\hat{a} \in C(\mathrm{Spec}(A))$ as in Theorem IV.5:

$$\alpha_{\Diamond}(J)(\hat{a}) = \widehat{\Diamond(a)}(J) \tag{7}$$

We can then uniquely extend $\alpha(J)$ to the whole space $C(\mathrm{Spec}(A))$ by using the fact that ϵ_A is an isometry with dense image. On a morphism $f : (A, u_A, \Diamond_A) \to (B, u_B, \Diamond_B)$ we define $\mathrm{Spec}^{\Diamond}(f)$ as $\mathrm{Spec}(f)$, where $\mathrm{Spec} : \mathbf{AURiesz}^{\mathrm{op}} \to \mathbf{CHaus}$ is the functor described in Section II-E.

The unit $\eta^{\Diamond} : \mathrm{id}_{\mathbf{Markov}} \to \mathrm{Spec}^{\Diamond} \circ C^{\Diamond}$ is defined exactly as the unit η from Section II-E. That is, for all Markov processes $\alpha : X \to \mathcal{R}^{\leq 1}(X)$, we define $\eta^{\Diamond}_{\alpha} = \alpha \overset{\eta_X}{\to} \mathrm{Spec}^{\Diamond}(C^{\Diamond}(\alpha))$ having underlying function $\eta_X : X \to \mathrm{Spec}(C(X))$.

Similarly, the counit $\epsilon^{\Diamond} : C^{\Diamond} \circ \mathrm{Spec}^{\Diamond} \to \mathrm{id}_{\mathbf{AURiesz}_{\Diamond}^{\mathrm{op}}}$ is defined exactly as the counit ϵ from Section II-E. That is, for $A = (A, u, \Diamond)$ in $\mathbf{AURiesz}_{\Diamond}$ we define $\epsilon^{\Diamond}_A = \epsilon_A$.

The fact that all previous definitions are consistent, e.g., that $C^{\Diamond}$ indeed maps coalgebra morphisms to modal Riesz space morphisms or that $\eta^{\Diamond}$ is indeed a collection of coalgebra morphisms, are summarized by the following theorem.

Theorem V.1. *As defined above, $C^{\Diamond}$ and $\mathrm{Spec}^{\Diamond}$ are functors and $\eta^{\Diamond}$ and $\epsilon^{\Diamond}$ are natural transformations. Furthermore $\mathrm{Spec}^{\Diamond}$ is a right adjoint to $C^{\Diamond}$, and restricts to an equivalence $\mathbf{Markov} \simeq \mathbf{CAURiesz}_{\Diamond}^{\mathrm{op}}$.*

VI. Initial Algebra and Final Coalgebra

In this section, exploiting the result of Theorem V.1, we study the properties of the final Markov process and of its dual object, the initial modal Riesz space in $\mathbf{CAURiesz}_{\Diamond}$.

A. Initial modal Riesz space

In this section we prove that the initial objects in the categories $\mathbf{Riesz}^u_{\Diamond}$ and its subcategory $\mathbf{AURiesz}^u_{\Diamond}$ exist and coincide. In other words, we show that the initial object in $\mathbf{Riesz}^u_{\Diamond}$ exists and is Archimedean and unital.

The following construction of the initial object in $\mathbf{Riesz}^u_{\Diamond}$ as a ground term algebra (free algebra of no generators) is standard. Consider the set $\mathtt{Form}$ of modal Riesz logic formulas (see Definition III.1), i.e., the set of terms build from the language $\{0, +, (r)_{r \in \mathbb{R}}, \sqcap, \sqcup, 1, \Diamond\}$. We define the equivalence relation $\equiv \subseteq \mathtt{Form} \times \mathtt{Form}$ as: $\phi \equiv \psi$ if and only if ϕ and ψ are provably equal (in equational logic) from the axioms of modal Riesz spaces when interpreting the atomic formula $1 \in \mathtt{Form}$ as the the constant u in the language of Modal Riesz spaces (Definition IV.1):

$$\phi \equiv \psi \quad \Longleftrightarrow \quad (\text{Axioms of } \mathbf{Riesz}^u_{\Diamond}) \vdash \phi = \psi$$

The collection of equivalence classes of $\equiv$ is denoted by $\mathbf{I}$

$$\mathbf{I} = \{ [\phi]_{\equiv} \mid \ \phi \text{ is a Riesz modal logic formula} \}$$

and is endowed with the structure of a modal Riesz space as follows: $\mathbb{I} = \langle \mathbf{I}, 0^{\mathbf{I}}, +^{\mathbf{I}}, (r^{\mathbf{I}})_{r \in \mathbb{R}}, \sqcap^{\mathbf{I}}, \sqcup^{\mathbf{I}}, u^{\mathbf{I}}, \Diamond^{\mathbf{I}} \rangle$ where:

$$0^{\mathbf{I}} = [0]_{\equiv} \qquad u^{\mathbf{I}} = [1]_{\equiv} \qquad \big([\phi]_{\equiv} +^{\mathbf{I}} [\psi]_{\equiv}\big) = [\phi + \psi]_{\equiv}$$
$$r^{\mathbf{I}}([\phi]_{\equiv}) = [r\phi]_{\equiv} \qquad \big([\phi]_{\equiv} \sqcap^{\mathbf{I}} [\psi]_{\equiv}\big) = [\phi \sqcap \psi]_{\equiv}$$
$$\big([\phi]_{\equiv} \sqcup^{\mathbf{I}} [\psi]_{\equiv}\big) = [\phi \sqcup \psi]_{\equiv} \qquad \Diamond^{\mathbf{I}}([\phi]_{\equiv}) = [\Diamond\phi]_{\equiv}$$

The following lemma is easy to check.

Lemma VI.1. $\mathbb{I} = \langle \mathbf{I}, 0^{\mathbf{I}}, +^{\mathbf{I}}, (r^{\mathbf{I}})_{r \in \mathbb{R}}, \sqcap^{\mathbf{I}}, \sqcup^{\mathbf{I}}, u^{\mathbf{I}}, \Diamond^{\mathbf{I}} \rangle$ *is a modal Riesz space and it is the initial object in the category $\mathbf{Riesz}^u_{\Diamond}$ of modal Riesz spaces.*

We then observe that the identified positive element $u^{\mathbf{I}} = [1]_{\equiv}$ (i.e., the set of formulas provably equivalent to the formula 1) is a strong unit of $\mathbb{I}$.

Theorem VI.2. *The element $u^{\mathbf{I}}$ is a strong unit of $\mathbb{I}$.*

Proof. We need to prove that for every formula ϕ, there exists some $n \in \mathbb{N}$ such that the inequality $|\phi| \leq n1$ is derivable by the axioms. This follows easily by induction on the structure of ϕ as follows. The base cases $\phi = 0$ and $\phi = 1$ are trivial. For the case $\phi = \phi_1 + \phi_2$ let us fix, using the inductive hypothesis, $n_1, n_2 \in \mathbb{N}$ such that $|\phi_1| \leq nu$ and $|\phi_2| \leq n_2 u$ respectively. Then the inequality $\phi_1 + \phi_2 \leq (n_1 + n_2)1$ is easily derivable. The cases for $\phi = r\phi_1$, $\phi = \phi_1 \sqcap \phi_2$ and $\phi = \phi_1 \sqcup \phi_2$ are similar. For the case $\phi = \Diamond\phi_1$, we can use the inductive hypothesis to get a number n_1 such that $|\phi_1| \leq n_1 1$. We will prove that $|\Diamond\phi_1| \leq (2n_1)1$.

From the hypothesis $|\phi_1| \leq n_1 1$ we can use the monotonicity of $\Diamond$ (see Remark IV.2), the linearity of $\Diamond$ and the the axiom $\Diamond(1) \leq 1$ to get:

$$\Diamond(|\phi_1|) \leq \Diamond(n_1 1) = n_1 \Diamond(1) \leq n_1 1$$

It then suffices to show that $|\Diamond(\phi_1)| \leq 2\Diamond(|\phi_1|)$. Recall that $|x| = x^+ + x^-$, $x^+ = x \sqcup 0$ and $x^- = (-x)^+$. Then using the linearity of $\Diamond$ we get:

$$\begin{aligned} |\Diamond(\phi_1)| &= \Diamond(\phi_1)^+ + \Diamond(\phi_1)^- \\ &= \Diamond(\phi_1)^+ + (-\Diamond(\phi_1))^+ \\ &= \Diamond(\phi_1)^+ + \Diamond(-\phi_1)^+ \end{aligned}$$

From the easily derivable (in)equalities $x \leq |x|$ and $|x| = |-x|$ and the monotonicity of $\Diamond$ we get:

$$\begin{aligned} \Diamond(\phi_1)^+ + \Diamond(-\phi_1)^+ &\leq \Diamond(|\phi_1|)^+ + \Diamond(|-\phi_1|)^+ \\ &= \Diamond(|\phi_1|)^+ + \Diamond(|\phi_1|)^+ \\ &= 2\Diamond(|\phi_1|)^+ \end{aligned}$$

By positivity of $\Diamond$ we have $\Diamond(|\phi_1|) \geq 0$ and therefore $\Diamond(|\phi_1|)^+ = \Diamond(|\phi_1|)$. Hence $|\Diamond(\phi_1)| \leq 2\Diamond(|\phi_1|)$ as desired. $\square$

Hence $(\mathbb{I}, u^{\mathbf{I}}, \Diamond^{\mathbf{I}}) \in \mathbf{URiesz}^u_{\Diamond}$. We can now invoke Theorem II.26 to prove the following result.

Theorem VI.3. *The modal Riesz space* $\mathbb{I}$ *is Archimedean.*

Proof. Since $u^{\mathbf{I}}$ is a strong unit it is sufficient, by Theorem II.26, to prove that for every nonzero element $[\phi]_{\equiv} \in \mathbb{I}$ there exists a Riesz homomorphism $f : (\mathbb{I}, u^{\mathbf{I}}) \to (\mathbb{R}, 1)$ which preserves the specified positive element (i.e., $f([1]_{\equiv}) = 1$) and such that $f([\phi]_{\equiv}) \neq 0$. Note that f is not required to preserve the $\Diamond$ operation. We construct f by first defining a map $g : \texttt{Form} \to \mathbb{R}$ defined inductively on the structure of formulas ϕ as follows:

$$\begin{gathered} g(0) = 0, \quad g(1) = 1, \quad g(r\phi) = rg(\phi) \\ g(\phi_1 + \phi_2) = g(\phi_1) + g(\phi_2) \\ g(\phi_1 \sqcup \phi_2) = \max\{g(\phi_1), g(\phi_2)\} \\ g(\phi_1 \sqcap \phi_2) = \min\{g(\phi_1), g(\phi_2)\} \\ g(\Diamond\phi) = g(\phi) \end{gathered}$$

We then let $f([\phi]_{\equiv}) = g(\phi)$ and it is easy to check that this is valid definition (i.e., if $\phi \equiv \psi$ then $g(\phi) = g(\psi)$) and that $f([1]_{\equiv}) = 1$. Furthermore, it is easy to prove by induction on the structure of ϕ that if $g(\phi) = 0$ then $[\phi]_{\equiv} = 0^{\mathbf{I}}$, i.e., ϕ is provably equal to 0. Hence, for every $[\phi]_{\equiv} \neq 0^{\mathbf{I}}$ we have $f([\phi]_{\equiv}) \neq 0$ and this completes the proof. □

The following is a direct consequence.

Corollary VI.4. *The modal Riesz space* $(\mathbb{I}, u^{\mathbb{I}}, \Diamond^{\mathbf{I}})$ *is initial in the category* $\mathbf{AURiesz}_{\Diamond}$ *of Archimedean and unital modal Riesz spaces.*

We now identify the initial object in the subcategory $\mathbf{CAURiesz}_{\Diamond}$ of uniformly complete Archimedean unital modal Riesz spaces. This is the the uniform completion (see Definition II.30) $C^{\Diamond}(\mathrm{Spec}^{\Diamond}((\mathbb{I})))$ of $\mathbb{I}$ which we denote by $\hat{\mathbb{I}}$.

Proposition VI.5. *The uniformly complete modal Riesz space* $\hat{\mathbb{I}}$ *is the the initial object in the category* $\mathbf{CAURiesz}_{\Diamond}$.

Proof. The functor $\mathrm{Spec}^{\Diamond}$, being a right adjoint, preserves limits, so it preserves terminal objects. Since $\mathbb{I}$ is initial in $\mathbf{AURiesz}_{\Diamond}$, this means that $\mathrm{Spec}^{\Diamond}(\mathbb{I})$ is the final coalgebra of $\mathbf{Markov}$. Since, restricted $\mathbf{CAURiesz}^{\mathrm{op}}_{\Diamond}$ the functor $C^{\Diamond}$ is an equivalence of categories, it does preserve terminal objects. Therefore $C^{\Diamond}(\mathrm{Spec}^{\Diamond}(\mathbb{I}))$ is the initial object of $\mathbf{CAURiesz}_{\Diamond}$ (terminal object of $\mathbf{CAURiesz}^{\mathrm{op}}_{\Diamond}$). The counit map $\epsilon^{\Diamond}_{\mathbb{I}}$ is such that $C^{\Diamond}(\mathrm{Spec}^{\Diamond}(\mathbb{I}))$ is isomorphic to the completion of $\mathbb{I}$. □

From Proposition II.31 we get that the two modal Riesz spaces $\mathbb{I}$ and $\hat{\mathbb{I}}$ are related by the following fact.

Proposition VI.6. *The modal Riesz space* $\mathbb{I}$ *embeds as a dense subalgebra of* $\hat{\mathbb{I}}$ *and the spectrums of* $\mathrm{Spec}(\mathbb{I})$ *and* $\mathrm{Spec}(\hat{\mathbb{I}})$ *are homeomorphic. In particular, there is a one-to-one correspondence between maximal ideals in* $\mathbb{I}$ *and* $\hat{\mathbb{I}}$.

B. Final Markov Process

We have identified in the previous section the initial object $\hat{\mathbb{I}}$ in the category $\mathbf{CAURiesz}_{\Diamond}$. From the duality between $\mathbf{CAURiesz}_{\Diamond}$ and $\mathbf{Markov}$ we can infer that the dual object of $\hat{\mathbb{I}}$ is the final coalgebra in $\mathbf{Markov}$. We denote this Markov process by $\alpha_{\mathbf{F}} : \mathbf{F} \to \mathcal{R}^{\leq 1}(\mathbf{F})$.

Recall, from Theorem IV.5, that its state space $\mathbf{F}$ is the compact Hausdorff space consisting of the collection of maximal ideals in $\hat{\mathbb{I}}$ endowed with the hull-kernel topology.

$$\mathbf{F} = \mathrm{Spec}(\hat{\mathbb{I}})$$

Therefore we can study the topological structure of the final coalgebra $\alpha_{\mathbb{F}}$ of $\mathbf{Markov}$ by studying the (hull-kernel topology of) the collection of maximal ideals in $\hat{\mathbb{I}}$. By Proposition VI.6, we can equivalently study the maximal ideals in $\mathbb{I}$.

$$\mathbf{F} = \mathrm{Spec}(\hat{\mathbb{I}}) = \mathrm{Spec}(\mathbb{I}) \tag{8}$$

To illustrate this method, consider the following Proposition VI.7 about the modal Riesz space $\mathbb{I}$. Recall from Definition II.21 that $\mathbb{I}$ is a normed space with norm $\|\_\|$ and compatible metric $d_{\mathbb{I}}$.

Proposition VI.7. *For each formula* ϕ *there exists a formula* ψ *having only rational coefficients such that* $d_{\mathbb{I}}([\phi]_{\equiv}, [\psi]_{\equiv}) \leq \epsilon$, *i.e., the inequality* $|\phi - \psi| \leq \epsilon 1$ *can be derived in equational logic form the axioms of modal Riesz spaces.*

Note that the set of formulas ψ having rational coefficients is countable. Hence $\mathbb{I}$ is *separable* as a metric space.

As a corollary of the previous proposition, which can be proved by elementary syntactical means such as induction over the structure of formulas, we get the following interesting property regarding the topology of $\mathbf{F}$.

Corollary VI.8. *The compact Hausdorff space* $\mathbf{F}$ *is second countable, i.e., there is a countable basis for the topology.*

As a known consequence of the Urysohn metrization theorem, every second countable compact Hausdorff space is Polish (i.e., separable complete metric space).

Corollary VI.9. $\mathbf{F}$ *is a Polish space.*

VII. Applications to the Riesz modal Logic

In this section we use the duality theorem and the characterization of the initial modal Riesz space $\hat{\mathbb{I}}$ and its dual, the final Markov process $\alpha_{\mathbb{F}}$, to prove basic results about the Riesz modal logic.

We start with the sound and complete axiomatization of the equivalence relation between modal Riesz logic formulas.

Theorem VII.1. *Let* $\phi, \psi \in \texttt{Form}$ *be two modal Riesz logic formulas. Then*

$$\phi \sim \psi \qquad \Longleftrightarrow \qquad (\text{Axioms of } \mathbf{Riesz}^u_{\Diamond}) \vdash \phi = \psi$$

Proof. Direction ($\Leftarrow$) (soundness). We know that to each Markov process $\alpha : X \to \mathcal{R}^{\leq 1}(X)$ corresponds the modal Riesz space $A_\alpha = (C(X), \mathbb{1}_X, \Diamond_\alpha)$ and, by Definition III.4, that $[\![\phi]\!]_\alpha = [\![\psi]\!]_\alpha$ holds if the equality $\phi = \psi$ holds in A_α. The assumption $(\text{Axioms of } \mathbf{Riesz}^u_{\Diamond}) \vdash \phi = \psi$ means that $\phi = \psi$ is true in all modal Riesz spaces and in particular in all A_α.

Direction ($\Rightarrow$) (completeness). Assume $\phi \sim \psi$ holds, i.e., the equality $[\![\phi]\!]_\alpha = [\![\psi]\!]_\alpha$ holds for all Markov processes $\alpha : X \to \mathcal{R}^{\leq 1}(X)$. In particular the equality holds for the final Markov process $\alpha_{\mathbb{F}} : \mathbf{F} \to \mathcal{R}^{\leq 1}(\mathbf{F})$. By duality, we have $C(\mathbf{F}) \simeq \hat{\mathbb{I}}$ and in particular, for all formulas $\phi \in \texttt{Form}$, we have the correspondence:

$$[\![\phi]\!]_{\alpha_{\mathbb{F}}} \longleftrightarrow [\phi]_{\equiv} \quad \text{and} \quad [\![\psi]\!]_{\alpha_{\mathbb{F}}} \longleftrightarrow [\psi]_{\equiv}$$

Therefore $[\![\phi]\!]_{\alpha_{\mathbb{F}}} = [\![\psi]\!]_{\alpha_{\mathbb{F}}}$ means $[\phi]_{\equiv} = [\psi]_{\equiv}$ or, equivalently, $\phi \equiv \psi$. By definition of the equivalence relation $\equiv$ this means:

$$(\text{Axioms of } \mathbf{Riesz}^u_\Diamond) \vdash \phi = \psi$$

and the proof is completed. □

The following theorem is another simple consequence of the machinery based on duality.

Theorem VII.2. *Let $x, y \in \mathbf{F}$ two points in the final coalgebra. If $x \neq y$ then there exists a formula ϕ such that that $[\![\phi]\!]_{\alpha_{\mathbb{F}}}(x) \neq [\![\phi]\!]_{\alpha_{\mathbb{F}}}(y)$.*

Proof. The space $\mathbf{F} = \mathrm{Spec}(\mathbb{I})$ is compact Hausdorff. Therefore points can be separated by continuous functions meaning that $x \neq y$ if and only if there exists a continuous function $f \in C(\mathbf{F})$ such that $f(x) \neq f(y)$. By duality we have that $C(\mathbf{F}) \simeq \hat{\mathbb{I}}$. Furthermore we know by Proposition VI.6 that $\mathbb{I}$ is a dense subalgebra of $\hat{\mathbb{I}}$. Hence, by choosing a sufficiently close approximation of f, we obtain a function $g \in \mathbb{I} \subseteq C(\mathbf{F})$ such that $g(x) \neq g(y)$. Now $g = [\phi]_{\equiv}$ for some formula $\phi \in \texttt{Form}$ and this is the desired separating formula. □

By combining Theorem VII.2 above with Proposition III.6 we then get the following corollary which states that modal Riesz logic formulas characterize behavioral equivalence. Recall that two states of a Markov process α are called *behaviorally equivalent* if the $\eta(x) = \eta(y)$ where $\alpha \stackrel{\eta}{\to} \alpha_{\mathbb{F}}$ is the unique coalgebra morphism from α to the final coalgebra.

Corollary VII.3. *Let $\alpha : X \to \mathcal{R}^{\leq 1}(X)$ be a Markov process and $x, y \in X$. Then x and y are behaviorally equivalent if and only if $[\![\phi]\!]_\alpha(x) = [\![\phi]\!]_\alpha(y)$ for all formulas ϕ.*

VIII. Conclusions

In this work we have laid the foundation of a theory of real–valued logics for Markov processes based on the theory of Riesz spaces. Many variations are possible. For example, one could consider *labeled* Markov processes [11] modeled as coalgebras of the functor $(\mathcal{R}^{\leq 1})^\Sigma$, for some set Σ of labels. One can then obtain a duality with multimodal Riesz spaces $(A, u, \{\Diamond_{a \in \Sigma}\})$. Alternatively, one could consider Markov processes as colagebras of the functor $\mathcal{R}^{=1} + 1$, modeling systems where at each state the computation either ends or progresses with probability 1. One can obtain a duality with modal Riesz spaces $(A, 1, \Diamond)$ whose $\Diamond$ operation is positive, linear, 1-preserving (i.e., $\Diamond(1) = 1$) and satisfies $\Diamond 1 = (\Diamond 1 + \Diamond 1) \sqcap 1$ (see Example III.13).

Among possible directions for future research, it would be interesting to extend this work from Markov processes to Markov decision processes (see [9, §4.1] for preliminary ideas in this direction). In another direction, it might be worth investigating the relation between the behavioral metrics for Markov processes studied in the literature and the metric space structure of the final Markov process (see Corollary VI.9). Lastly, one can investigate an extension of the Riesz modal logic with operators for (co)inductive definitions. This would allow the interpretation of the full Łukasiewicz modal μ–calculus and therefore also of pCTL.

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Appendix

This appendix contains the proofs of Theorems II.28 and II.29, Propositions III.6 and VI.7 and Corollary VI.8.

Theorem A.1. Spec *is a right adjoint to* C*, with unit* $\eta_X = N_{\text{-}}$ *and counit* $\epsilon_A = \hat{\text{-}}$*. When restricted to* $\mathbf{CAURiesz}^{\mathrm{op}}$*,* C *and* Spec *form an adjoint equivalence. An object* (A, u) *of* $\mathbf{AURiesz}$ *is uniformly complete (i.e., belong to* $\mathbf{CAURiesz}$*) if and only if the counit of the adjunction is an isomorphism.*

Proof. The fact that C is indeed a functor follows from elementary properties of composition of functions and identity maps. We now show Spec is a functor. Let $f : (A, u_A) \to (B, u_B)$ be a unital Riesz homomorphism and $J \subseteq B$ a maximal ideal. By Theorem II.25 there is a unital Riesz morphism $\phi_J : B \to \mathbb{R}$ such that $\phi_J^{-1}(0) = J$. The composite $\phi_J \circ f$ is a unital Riesz homomorphism $A \to \mathbb{R}$, so

$$\mathrm{Spec}(f)(J) = f^{-1}(J) = (\phi_J \circ f)^{-1}(0)$$

is a maximal ideal in A. This shows $\mathrm{Spec}(f)$ is a function $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$. We show it is continuous by showing that the preimage of a closed set is closed. Any closed set in $\mathrm{Spec}(A)$ is $\mathrm{hull}(I)$ for some ideal $I \subseteq A$. By [12, Theorem 59.2 (iii)], $f(I)$ is also an ideal. By elementary manipulations of the definitions, $\mathrm{Spec}(f)^{-1}(\mathrm{hull}(I)) = \mathrm{hull}(f(I))$, which, as we started with an arbitrary closed set, proves the continuity of $\mathrm{Spec}(f)$. By basic properties of the preimage mapping, Spec preserves identity maps and reverses composition, and is therefore a contravariant functor, as required.

We now consider the unit. In [26, Theorem 4] Yosida shows that the mapping $N_{\text{-}} : X \to \mathrm{Spec}(C(X))$ is a homeomorphism onto its image and has dense image. The compactness of X then implies that the image of $N_{\text{-}}$ is closed, and therefore is all of $\mathrm{Spec}(C(X))$, *i.e.*, $N_{\text{-}}$ is a homeomorphism $X \to \mathrm{Spec}(C(X))$.

The proof of naturality, *i.e.*, that $N_{f(x)} = \mathrm{Spec}(C(f))(N_x)$ for all continuous maps $f : X \to Y$ and for all $x \in X$, is done by expanding the definitions on each side, so is omitted. As a result, $\mathrm{Spec} \circ C \cong \mathrm{Id}_{\mathbf{CHaus}}$.

For the counit, Yosida shows that $\hat{a} \in C(\mathrm{Spec}(A))$, and $\hat{\text{-}}$ is a unital Riesz space homomorphism with norm-dense image, and an isomorphism iff A is uniformly complete [26, Theorems 1–3] (see also [12, Theorems 45.3 and 45.4]). As the (norm) unit ball of A is exactly the inverse image of the unit ball of $C(\mathrm{Spec}(A))$, we have that the embedding $\hat{\text{-}}$ is also an isometry, and therefore $C(\mathrm{Spec}(A))$ is isomorphic to the Banach space completion of A.

The naturality of $\hat{\text{-}}$, *i.e.*, that for all $f : A \to B$ a unital Riesz homomorphism, $a \in A$, $J \in \mathrm{Spec}(B)$ we have $\widehat{f(a)}(J) = C(\mathrm{Spec}(f))(\hat{a})(J)$ reduces to showing that $\hat{a}(f^{-1}(J)) \cdot u_A - f(a) \in I$, which is easily done using the linearity and unitality of f and the definition of $\hat{\text{-}}$.

To show that Spec is a right adjoint to C, we only need to prove that the following diagrams commute:

$$\begin{tikzcd}
C(X) & C(\mathrm{Spec}(C(X))) \arrow[l, "C(\eta_X)"'] \\
& C(X) \arrow[u, "\epsilon_{C(X)}"'] \arrow[ul, "\mathrm{id}"]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Spec}(A) \arrow[r, "\eta_{\mathrm{Spec}(A)}"] \arrow[dr, "\mathrm{id}"'] & \mathrm{Spec}(C(\mathrm{Spec}(A))) \arrow[d, "\mathrm{Spec}(\epsilon_A)"] \\
& \mathrm{Spec}(A),
\end{tikzcd}$$

where X is a compact Hausdorff space and A a unital Archimedean Riesz space.

For the first diagram, we want to show that if $a \in C(X)$ and $x \in X$, we have $C(\eta_X)(\epsilon_{C(X)}(a))(x) = a(x)$. Expanding the definitions shows us that this is equivalent to showing

$$\hat{a}(N_x) = a(x). \tag{9}$$

By the definition of $\hat{\text{-}}$, we have $\hat{a}(N_x) \cdot \mathbb{1}_X - a \in N_x$. Applying the definition of N_x and elementary algebra then gives us the result.

For the second diagram, we want to show that for each $J \in \mathrm{Spec}(A)$ and $a \in A$ that $a \in \mathrm{Spec}(\epsilon_A)(\eta_{\mathrm{Spec}(A)}(J)) \Leftrightarrow a \in J$. This can be proved simply by expanding the definitions.

We have therefore shown that $C \dashv \mathrm{Spec}$, *i.e.*, Spec is a right adjoint to C. We already saw that Yosida proved that η_X is always an isomorphism for X a compact Hausdorff space, and that ϵ_A is an isomorphism iff A is uniformly complete. Therefore $(C, \mathrm{Spec}, \eta, \epsilon)$ is an adjoint equivalence when restricted to $\mathbf{CAURiesz}$ [25, §IV.4]. ∎

As any unital Archimedean Riesz space (A, u_A) embeds densely and isometrically in $C(\mathrm{Spec}(A))$, when using their natural norms (Theorem II.21), we have that $\hat{\text{-}}$ extends to an isomorphism between the norm completion of A and $C(\mathrm{Spec}(A))$. In fact

Corollary A.2. $C(\mathrm{Spec}(A))$ *is isomorphic to the norm completion of* A*, and any unital Riesz homomorphism* $f : (A, u_A) \to (B, u_B)$ *extends to a unique unital Riesz homomorphism between their norm completions.*

The extension in the above corollary can be seen to be a unital Riesz homomorphism because $C(\mathrm{Spec}(f))$ is one, and $\hat{\text{-}}$ is a natural transformation. The extension is unique because continuous functions that agree on a dense subset are equal.

The following is a proof of Theorem V.1.

Theorem A.3. *As defined in Section V,* $C^{\Diamond}$ *and* $\mathrm{Spec}^{\Diamond}$ *are functors and* $\eta^{\Diamond}$ *and* $\epsilon^{\Diamond}$ *are natural transformations. Furthermore* $\mathrm{Spec}^{\Diamond}$ *is a right adjoint to* $C^{\Diamond}$*, and restricts to an equivalence* $\mathbf{Markov} \simeq \mathbf{CAURiesz}_{\Diamond}^{\mathrm{op}}$.

Proof. By Example IV.4 we have that if (X, α) is a Markov process, $(C(X), \Diamond_\alpha)$ is an object of $\mathbf{CAURiesz}_{\Diamond}$. We now show that if $f : X \to Y$ underlies a Markov process homomorphism $(X,\alpha) \to (Y,\beta)$, then $C^{\Diamond}(f)$ is a morphism in $\mathbf{CAURiesz}_{\Diamond}$ from $C(Y) \to C(X)$, *i.e.* if the diagram (3) commutes, then $C(f) \circ \Diamond_\beta = \Diamond_\alpha \circ C(f)$, as follows. We prove this by applying the left hand side to arbitrary elements $b \in C(Y)$ and $x \in X$:

$$\begin{aligned} C(f)(\Diamond_\beta(b))(x) &= \Diamond_\beta(b)(f(x)) \\ &= \beta(f(x))(b) && \text{by (6)} \\ &= \mathcal{R}^{\leq 1}(f)(\alpha(x))(b) && \text{by (3)} \\ &= \alpha(x)(b \circ f) && \text{by (2)} \\ &= \alpha(x)(C(f)(b)) \\ &= \Diamond_\alpha(C(f)(b))(x) && \text{by (6).} \end{aligned}$$

We then have that $C^{\Diamond}$ preserves the identity maps and composition because C does so as a functor $\mathbf{CHaus} \to \mathbf{CAURiesz}^{\mathrm{op}}$.

We show that, for $(A,u,\Diamond) \in \mathbf{AURiesz}_{\Diamond}$, $(\mathrm{Spec}(A), \alpha_\Diamond)$ is a Markov process as follows. By the pointwiseness of the definitions, $\alpha_\Diamond(J)$ is positive and unital for all $J \in \mathrm{Spec}(A)$, at least on the subspace $\hat{-}(A) \subseteq C(\mathrm{Spec}(A))$. Its extension to $C(\mathrm{Spec}(A))$ is positive and unital because the positive cone in $C(\mathrm{Spec}(A))$ is (norm) closed. We show that $\alpha_\Diamond$ is a continuous map as follows. Let $(J_i)_{i\in I}$ be a net converging to an ideal J in the hull-kernel topology of $\mathrm{Spec}(A)$. For each $a \in A$, we have that

$$\begin{aligned} \alpha_\Diamond\left(\lim_i J_i\right)(\hat{a}) &= \widehat{\Diamond(a)}\left(\lim_i J_i\right) = \lim_i \widehat{\Diamond(a)}(J_i) \\ &= \lim_i \alpha_\Diamond(J_i)(\hat{a}). \end{aligned}$$

Therefore $\alpha_\Diamond$ is continuous in the weak-* topology defined by $\hat{-}(A) \subseteq C(\mathrm{Spec}(A))$. As $\hat{-}(A)$ is dense in $C(\mathrm{Spec}(A))$ and $\mathcal{R}^{\leq 1}(\mathrm{Spec}(A)) \subseteq C(\mathrm{Spec}(A))^*$ is norm-bounded and therefore equicontinuous, this topology agrees with the usual weak-* topology defined by $C(\mathrm{Spec}(A))$ on $\mathcal{R}^{\leq 1}(\mathrm{Spec}(A))$ [28, III.4.5]. Therefore $\mathrm{Spec}(A,u,\Diamond)$ is always a Markov process.

Let $f : (A,u_A,\Diamond_A) \to (B,u_B,\Diamond_B)$ be a morphism in $\mathbf{AURiesz}_{\Diamond}$. We want to show that $\mathrm{Spec}^{\Diamond}(f)$ is a morphism of Markov processes, *i.e.*, $\alpha_{\Diamond_A} \circ \mathrm{Spec}(f) = \mathcal{R}^{\leq 1}(\mathrm{Spec}(f)) \circ \alpha_{\Diamond_B}$. We do this by proving that for all $J \in \mathrm{Spec}(B)$ and $a \in A$ that $\alpha_{\Diamond_A}(\mathrm{Spec}(f)(J))(\hat{a}) = \mathcal{R}^{\leq 1}(\mathrm{Spec}(f))(\alpha_{\Diamond_B}(J))(\hat{a})$, using the denseness of $\hat{-}(A) \subseteq C(\mathrm{Spec}(A))$. We have, writing "nat" to indicate the use of the naturality of $\hat{-}$ from Theorem II.29,

$$\begin{aligned} \mathcal{R}^{\leq 1}(\mathrm{Spec}(f))(\alpha_{\Diamond_B}(J))(\hat{a}) &= \alpha_{\Diamond_B}(J)(C(\mathrm{Spec}(f))(\hat{a})) \\ &= \alpha_{\Diamond_B}(J)(\widehat{f(a)}) && \text{nat} \\ &= \widehat{\Diamond_B(f(a))}(J) \\ &= \widehat{f(\Diamond_A(a))}(J) \\ &= C(\mathrm{Spec}(f))(\widehat{\Diamond_A(a)})(J) && \text{nat} \\ &= \widehat{\Diamond_A(a)}(\mathrm{Spec}(f)(J)) \\ &= \alpha_{\Diamond_A}(\mathrm{Spec}(f)(J))(\hat{a}). \end{aligned}$$

As in the case of $C^{\Diamond}$, the rest of the proof that $\mathrm{Spec}^{\Diamond}$ is a functor follows as in Theorem II.29 from the fact that Spec is a functor.

We can finish the proof that this is a dual adjunction that restricts to a duality $\mathbf{CAURiesz}_{\Diamond}^{\mathrm{op}} \simeq \mathbf{Markov}$ by proving that N_{-} and $\hat{-}$, the unit and counit of the adjunction in Theorem II.29, are a morphism of Markov processes and a modal Riesz homomorphism, respectively. The reason for this is that diagrams in $\mathbf{Markov}$ (respectively, in $\mathbf{AURiesz}_{\Diamond}$) commute iff their underlying diagrams in $\mathbf{CHaus}$ (respectively, in $\mathbf{AURiesz}$) commute, and morphisms in $\mathbf{Markov}$ (respectively, in $\mathbf{AURiesz}_{\Diamond}$) are isomorphisms iff their underlying morphisms in $\mathbf{CHaus}$ (respectively, in $\mathbf{AURiesz}$) are isomorphisms.

We first show that $\hat{-}$ is a modal Riesz homomorphisms, *i.e.*, that if $(A,u,\Diamond)$ is an object of $\mathbf{AURiesz}_{\Diamond}$, $\widehat{\Diamond(a)} = \Diamond_{\alpha_\Diamond}(\hat{a})$ for all $a \in A$. Let $J \in \mathrm{Spec}(A)$:

$$\Diamond_{\alpha_\Diamond}(\hat{a})(J) = \alpha_\Diamond(J)(\hat{a}) = \widehat{\Diamond(a)}(J).$$

Now we want to show N_{-} is a Markov morphism, *i.e.*, that if (X,α) is a Markov process, $\mathcal{R}^{\leq 1}(N_{-}) \circ \alpha = \alpha_{\Diamond_\alpha} \circ N_{-}$. We use the fact that each $b \in C(\mathrm{Spec}(C(X)))$ is of the form $b = \hat{a}$ for some $a \in C(X)$ (Theorem II.29) to reduce this to showing that $\mathcal{R}^{\leq 1}(N_{-})(\alpha(x))(\hat{a}) = \alpha_{\Diamond_\alpha}(N_x)(\hat{a})$. Observe that

$$\begin{aligned} \alpha_{\Diamond_\alpha}(N_x)(\hat{a}) &= \widehat{\Diamond_\alpha(a)}(N_x) \\ &= \Diamond_\alpha(a)(x) && (9) \\ &= \alpha(x)(a) \\ &= \alpha(x)(\hat{a} \circ N_{-}) && (9) \\ &= \mathcal{R}^{\leq 1}(N_{-})(\alpha(x))(\hat{a}). \end{aligned}$$

This concludes the proof. ☐

The following is a proof of Proposition III.6.

Proposition A.4. *Let $\alpha : X \to \mathcal{R}^{\leq 1}(X)$ and $\beta : Y \to \mathcal{R}^{\leq 1}(Y)$ be two Markov processes and let $\alpha \xrightarrow{f} \beta$ be a coalgebra morphism. For every formula ϕ the equality $[\![\phi]\!]_\alpha = [\![\phi]\!]_\beta \circ f$ holds, i.e., $[\![\phi]\!]_\alpha(x) = [\![\phi]\!]_\beta(f(x))$, for all $x \in X$.*

Proof. We simply need to unfold the definitions. Recall that a coalgebra morphism $\alpha \xrightarrow{f} \beta$ is a continuous function $f : X \to Y$ such that $\beta(f(x)) = \mathcal{R}^{\leq 1}(f)\big(\alpha(x)\big)$ holds. By definition of the action of the Radon functor $\mathcal{R}^{\leq 1}$ on morphisms (see Section II-A) we have that the probability measure $\beta(f(x))$, or equivalently its corresponding expectation functional $\mathbb{E}_{\beta(f(x))} : C(Y) \to \mathbb{R}$, is definable as follows:

$$\mathbb{E}_{\beta(f(x))}(b) = \mathbb{E}_{\alpha(x)}(b \circ f) \tag{10}$$

for all $b \in C(Y)$. We prove the statement $[\![\phi]\!]_\alpha = [\![\phi]\!]_\beta \circ f$ by induction on the structure of ϕ. The only non trivial case is that of $\phi = \Diamond\psi$. By definition we have:

$$[\![\Diamond\psi]\!]_\alpha(x) = \mathbb{E}_{\alpha(x)}([\![\psi]\!]_\alpha) \text{ and } [\![\Diamond\psi]\!]_\beta(f(x)) = \mathbb{E}_{\beta(f(x))}([\![\psi]\!]_\beta)$$

Therefore, by Equation 10 above, we obtain the equality $[\![\Diamond\psi]\!]_\beta(f(x)) = \mathbb{E}_{\alpha(x)}([\![\psi]\!]_\beta \circ f)$. The inductive hypothesis $[\![\psi]\!]_\alpha = [\![\psi]\!]_\beta \circ f$ on ψ then concludes the proof. □

The following is a Proof of Proposition VI.7.

Proposition A.5. *For each formula ϕ there exists a formula ψ having only rational coefficients such that $d_{\mathbb{I}}([\phi]_\equiv, [\psi]_\equiv) \leq \epsilon$, i.e., the inequality $|\phi - \psi| \leq \epsilon 1$ can be derived in equational logic form the axioms of modal Riesz spaces.*

Proof. The proof goes by induction on the modal-depth $m(\phi)$ of ϕ defined inductively by $m(0) = m(1) = 0$, $m(\Diamond\phi) = 1 + m(\phi)$ and $m(\phi_1 + \phi_2) = \max\{m(\phi_1), m(\phi_2)\}$ and similarly for all other connectives. The base case $m(\phi) = 0$ is trivial, as ϕ can be identified with a real number $r_\phi \in \mathbb{R}$ and we can choose ψ to be $s1$ for some rational s such that $|r - s| < \epsilon$.

Suppose now that $m(\phi) = k + 1$. We have to consider all separate cases. The most interesting is the case $\phi = \Diamond\phi_1$. We can pick, by inductive hypothesis, a formula ψ with rational coefficients such that $[|\phi - \psi|]_\equiv < [\frac{\epsilon}{2} 1]_\equiv$. Then, using the same kind of computations described in the proof of Theorem VI.2 based on linearity, positivity and 1-decreasing axioms of $\Diamond$, that $|\Diamond\phi - \Diamond\psi| < [\epsilon 1]_\equiv$. All other cases involving the other connectives follow easily form the inductive hypothesis. For example, if $\phi = \phi_1 + \phi_2$ then, by inductive hypothesis, we can pick ψ_1 and ψ_2 such that $|\phi_i - \psi_i|_\equiv < \frac{1}{2}\epsilon[1]_\equiv$. It then clearly follows that $\psi = \psi_1 + \psi_2$ has the required property. □

Note that the set of formulas ψ having rational coefficients is countable. Hence $\mathbb{I}$ is *separable* as a metric space.

The following is a proof of Corollary VI.8.

Corollary A.6. *The compact Hausdorff space* **F** *is second countable, i.e., there is a countable basis for the topology.*

Proof. In what follows, to simplify the notation, we simply write ϕ in place of $[\phi]_\equiv \in \mathbb{I}$. We reserve the letter ψ to range over formulas with rational coefficients.

A basis for the closed sets of **F** is (see Section 35 of [12]) the collection $C_\phi = \{J \in \mathbf{F} \mid \phi \in J\}$ of sets of maximal ideals in $\mathbb{I}$ containing the positive element $\phi > 0$. We show that the countable sub-collection C_ψ is also a basis by proving that

$$C_\phi = \bigcap \{C_\psi \mid \psi < |\phi|\} \ .$$

The inclusion $C_\phi \subseteq \bigcap_\psi C_\psi$ follows by a basic property of ideals: $a \in J$ and $b \leq |a|$ implies $b \in J$.

So assume towards a contradiction that $C_\phi \subsetneq \bigcap_\psi C_\psi$. This means that there exists a maximal ideal $J \in \bigcap_\psi C_\psi$ such that $J \notin C_\phi$. This means:

1) for all $\psi < |\phi|$ it holds that $\psi \in J$, and
2) $\phi \notin J$.

We show that this is impossible by finding a ψ such that $\psi \notin J$.

Let $f_J : (\mathbb{I}, u) \rightarrow (\mathbb{R}, 1)$ be the unique unital Riesz homomorphism such that $f^{-1}(0) = J$ (see Theorem II.25). We have that

1) for all $\psi < |\psi|$ it holds that $f(\psi) = 0$, and
2) $f(\phi) = r$, for some $r > 0$.

Using Proposition VI.7, pick a formula ψ with rational coefficients such that $\|\phi - \psi\| \leq \epsilon$, for some $0 < \epsilon < r$. This means that $|\phi - \psi| \leq \frac{\epsilon}{1}$ holds in $\mathbb{I}$.

Therefore by monotonicity of the unital homomorphism f, we have $f(|\phi - \psi|) \leq \epsilon$. As the the equality $f(|a|) = |f(a)|$ holds for arbitrary homomorphisms of Riesz spaces, we have:

$$|f(\phi - \psi)| = |f(\phi) - f(\psi)| \leq \epsilon$$

But this provides the desired contradiction as $f(\phi) = r$, $f(\psi) = 0$ and $\epsilon < r$. □