Undecidability of Domino Games and Hhp-Bisimilarity

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History preserving bisimilarity (hp-bisimilarity) and hereditary history preserving bisimilarity (hhp-bisimilarity) are behavioural equivalences taking into account causal relationships between events of concurrent systems. Their prominent feature is that they are preserved under action refinement, an operation important for the top-down design of concurrent systems. It is shown that, in contrast to hp-bisimilarity, checking hhp-bisimilarity for finite labelled asynchronous transition systems is undecidable, by a reduction from the halting problem of 2-counter machines. To make the proof more transparent a novel intermediate problem of checking domino bisimilarity for origin constrained tiling systems is introduced and shown undecidable.

It is also shown that the unlabelled domino bisimilarity problem is undecidable, which implies undecidability of hhp-bisimilarity for unlabelled finite asynchronous systems. Moreover, it is argued that the undecidability of hhp-bisimilarity holds for finite elementary net systems.

1. INTRODUCTION

The notion of behavioural equivalence which has attracted most attention in concurrency theory is bisimilarity, originally introduced by Park [25] and Milner [19]; concurrent programs are considered to have the same meaning if they are bisimilar. The prominent role of bisimilarity is due to many pleasant properties it enjoys; we mention a few of them here.

A process of checking whether two transition systems are bisimilar can be seen as a two player game which is in fact an Ehrenfeucht-Fraïssé type of game for modal logic. More precisely, there is a winning strategy for a player who wants to show

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that the systems are bisimilar if and only if the systems cannot be distinguished by
the formulas of the logic; the result due to Hennessy and Milner [12].

Another notable property of bisimilarity is its computational feasibility; see for
example the overview note [21]. Let us illustrate this on the examples of finite
transition systems and a class of infinite-state transition systems generated by con-
text free grammars. For finite transition systems there are very efficient polynomial
time algorithms for checking bisimilarity [17, 24], in sharp contrast to PSPACE-
completeness of the classical language equivalence. For transition systems generated
by context free grammars, while language equivalence is undecidable, bisimilarity
is decidable [4], and if the grammar has no redundant nonterminals, even in poly-
nomial time [13]. Furthermore, as the results of Groote and Hüttel [11] indicate,
bisimilarity has a very rare status of being a decidable equivalence for context free
grammars: all the other equivalences in the linear-branching time hierarchy [8] are
undecidable. The algorithmic tractability makes bisimilarity especially attractive
for automatic verification of concurrent systems.

The essence of bisimilarity, quoting Hennessy and Milner [12], “is that the be-

haviour of a program is determined by how it communicates with an observer.”
Therefore, the notion of what can be observed of a behaviour of a system affects
the notion of bisimilarity. An abstract definition of bisimilarity for arbitrary cat-

ergories of models due to Joyal et al. [15] formalizes this idea. Given a category

dead of models where objects are behaviours and morphisms correspond to extension of
behaviours, and given a subcategory of observable behaviours, the abstract de-

finition yields a notion of bisimilarity for all behaviours with respect to observable
behaviours. For example, for rooted labelled transition systems, taking synchro-
nization trees [19] into which they unfold as their behaviours, and sequences of
actions as the observable behaviours, we recover the standard strong bisimilarity
of Park and Milner [15].

In order to model concurrency more faithfully several models have been intro-
duced (see [30] for a survey) that make explicit the distinction between events that
can occur concurrently, and those that are causally related. Then a natural choice
is to replace sequences, i.e., linear orders as the observable behaviours, by partial
orders of occurrences of events with causality as the ordering relation. For example,
taking unfoldings of labelled asynchronous transition systems into event structures
as the behaviours, and labelled partial orders as the observations, Joyal et al. [15]

obtained from their abstract definition the hereditary history preserving bisimilarity
(hhp-bisimilarity), independently introduced and studied by Bednarczyk [1].

A similar notion of bisimilarity has been studied before, namely history preserving
bisimilarity (hp-bisimilarity), introduced by Rabinovich and Trakhtenbrot [26] and
van Glabbeek and Goltz [9]. For the relationship between hp- and hhp-bisimilarity
see for example [1, 15, 7].

One of the important motivations to study partial order based equivalences was
the discovery that hp-bisimilarity has a rare status of being preserved under ac-
tion refinement [9], an operation important for the top-down design of concurrent

systems. Bednarczyk [1] has extended this result to hhp-bisimilarity.

There is a natural logical characterization of hhp-bisimilarity checking games as
shown by Nielsen and Clausen [22]: they are characteristic games for an extension
of modal logic with backwards modalities, interpreted over event structures.
Hp-bisimilarity has been shown to be decidable for 1-safe Petri nets by Vogler [29], and to be DEXP-complete by Jategaonkar, and Meyer [14]; let us just mention here that 1-safe Petri nets can be regarded as a proper subclass of finite asynchronous transition systems (see [30] for details), and that decidability of hp-bisimilarity can be easily extended to all finite asynchronous transition systems using the methods of [14].

Hp-bisimilarity appears to be only a slight strengthening of hp-bisimilarity [15] and hence many attempts have been made to extend the above mentioned algorithms to the case of hhp-bisimilarity. However, decidability of hhp-bisimilarity has remained open, despite several attempts over the years [22, 23, 3, 7]. Fröschle and Hildebrandt [7] have discovered an infinite hierarchy of bisimilarity notions refining hp-bisimilarity, and coarser than hhp-bisimilarity, such that hhp-bisimilarity is the intersection of all the bisimilarities in the hierarchy. They have shown all these bisimilarities to be decidable for 1-safe Petri nets. Fröschle [6] has proved hhp-bisimilarity to be decidable for BPP-processes, a class of infinite state systems.

In this paper we resolve the question of decidability of hhp-bisimilarity for all finite state systems by showing it to be undecidable for finite, both labelled and unlabelled, asynchronous transition systems and finite elementary net systems. In order to make the proof more transparent we first introduce an intermediate problem of domino bisimilarity and show its undecidability by a direct reduction from the undecidable halting problem for 2-counter machines [20]. The undecidability of the novel problem of checking domino bisimilarity seems to be interesting in its own right and does not follow from somewhat related results for domino snakes [5] and domino games [10], nor from the undecidability of the classical tiling problems [2].

2. HEREDITARY HISTORY PRESERVING BISIMILARITY

In this section we define hereditary history preserving bisimulation for asynchronous transition systems and we introduce the algorithmic problem of checking hereditary history preserving bisimilarity. We also mention the equivalent, and sometimes technically more convenient, problem of solving hereditary history preserving bisimilarity checking games.

Definition 2.1. (Labelled/unlabelled asynchronous transition system). A labelled asynchronous transition system is a tuple $A = (S, s^{\text{ini}}, E, \rightarrow, L, \lambda, I)$, where $S$ is its set of states, $s^{\text{ini}} \in S$ is the initial state, $E$ is the set of events, $\rightarrow \subseteq S \times E \times S$ is the set of transitions, $L$ is the set of labels, and $\lambda : E \rightarrow L$ is the labelling function, and $I \subseteq E^2$ is the independence relation which is irreflexive and symmetric. We often write $s \xrightarrow{e} s'$, instead of $(s, e, s') \in \rightarrow$. Moreover, the following conditions have to be satisfied:

1. if $s \xrightarrow{e} s'$ and $s \xrightarrow{e'} s''$ then $s' = s''$,
2. if $(e, e') \in I$, $s \xrightarrow{e} s'$, and $s' \xrightarrow{e'} t$, then $s \xrightarrow{e'} s''$, and $s'' \xrightarrow{e} t$ for some $s'' \in S$. 

An asynchronous transition system is coherent if it satisfies the following condition:

3. if $(e, e') \in I$, $s \xrightarrow{e} s'$, and $s' \xrightarrow{e'} s''$, then $s' \xrightarrow{e'} t$, and $s'' \xrightarrow{e} t$ for some $t \in S$.

An asynchronous transition system is prime if it is acyclic and satisfies the following condition:
4. if \( s \xrightarrow{\epsilon} t \) and \( s' \xrightarrow{\epsilon'} t \) then \((\epsilon, \epsilon') \in I\).

We say that an asynchronous transition system is \textit{unlabelled} if the set of labels \( L \) is a singleton set.

Winskel and Nielsen [30, 23] give a thorough survey and establish formal relationships between asynchronous transition systems and other models for concurrency, such as Petri nets, and event structures. The independence relation is meant to model concurrency: independent events can occur concurrently, while those that are not independent are causally related or in conflict.

Let \( A = (S, s^\text{ini}, E, \rightarrow, L, \lambda, I) \) be a labelled asynchronous transition system. A sequence of events \( \tau = (e_1, e_2, \ldots, e_n) \in E^* \) is a run of \( A \) if there are states \( s_1, s_2, \ldots, s_n + 1 \in S \), such that \( s_1 = s^\text{ini} \), and for all \( i \in \{1, 2, \ldots, n\} \), we have \( s_i \xrightarrow{e_i} s_{i+1} \). We write \( \text{Runs}(A) \) to denote the set of runs of \( A \). We extend the labelling function \( \lambda \) to runs in the standard way.

Let \( \tau = (e_1, e_2, \ldots, e_n) \in \text{Runs}(A) \). We say that the \( k \)-th event, \( 1 \leq k < n \), is \textit{swappable} in \( \tau \) if \((e_k, e_{k+1}) \in I\). We define \( \text{Swap}(\tau) \) to be the set of numbers of swappable events in \( \tau \). We write \( \tau \otimes k \) to denote the result of \textit{swapping} the \( k \)-th event of \( \tau \) with the \((k + 1)\)-st, i.e., the sequence \((e_1, \ldots, e_{k-1}, e_{k+1}, e_k, \ldots, e_n)\). Note that if \( k \in \text{Swap}(\tau) \) then \( \tau \otimes k \in \text{Runs}(A) \); it follows from condition 2. of the definition of an asynchronous transition system.

A run of a transition system models a finite \textit{sequential} behaviour of a system: a sequence of occurrences of events. In order to model \textit{concurrent} behaviours of a system we define an equivalence relation on the set of runs of an asynchronous transition system. We define the equivalence relation \( \equiv_A \) on \( \text{Runs}(A) \) to be the reflexive, symmetric, and transitive closure of

\[
\{ (\tau, \tau \otimes k) : \tau \in \text{Runs}(A) \text{ and } k \in \text{Swap}(\tau) \}.
\]

In other words, we have that \( \tau_1 \equiv_A \tau_2 \), for \( \tau_1, \tau_2 \in \text{Runs}(A) \), if and only if \( \tau_2 \) can be obtained from \( \tau_1 \) by a finite number of swaps of swappable events.

We define an unfolding operation on asynchronous transition systems into prime asynchronous transition systems. The states of the unfolding of an asynchronous transition system \( A \) are meant to represent all concurrent behaviours of a system, just like the states of a synchronisation tree represent all sequential behaviours of a system.

**Definition 2.2. (Unfolding).**
Let \( A = (S, s^\text{ini}, E, \rightarrow, L, \lambda, I) \) be an asynchronous transition system. The unfolding \( \text{Unf}(A) \) of \( A \) is an asynchronous transition system with the same set of events, the labelling function, and the independence relation as \( A \). The set of states, the initial state, and the transition relation of \( \text{Unf}(A) \) are defined as follows:

- the set of states \( S_{\text{Unf}(A)} \) of \( \text{Unf}(A) \) is defined to be \( \text{Runs}(A) / \equiv_A \), i.e., the set of concurrent behaviours of \( A \),
- the initial state \( s^\text{ini}_{\text{Unf}(A)} \) of \( \text{Unf}(A) \) is \([\epsilon]_{\equiv_A} \), i.e., the \( \equiv_A \)-equivalence class of the empty run,
- the set of transitions \( \rightarrow_{\text{Unf}(A)} \) of \( \text{Unf}(A) \) consists of transitions of the form \(([\tau])_{\equiv_A}, \epsilon, ([\tau \cdot \epsilon])_{\equiv_A} \) for all \( \tau \in E^* \), and \( \epsilon \in E \), such that \( \tau \cdot \epsilon \in \text{Runs}(A) \).
The following proposition follows easily from the definition of $\Unf(A)$.

**Proposition 2.1.** If $A$ is an asynchronous transition system then its unfolding $\Unf(A)$ is a prime asynchronous transition system.

Let $\tau = \langle e_1, e_2, \ldots, e_n \rangle \in \text{Runs}(A)$. We say that the $k$-th event, $1 \leq k \leq n$, is most recent in $\tau$ if and only if $(e_k, e_\ell) \in I$, for all $\ell$, such that $k < \ell \leq n$. We define $\text{MR}(\tau)$ to be the set of numbers of most recent events in $\tau$. We write $\tau \odot k$ to denote the result of removing the $k$-th event from $\tau$, i.e., the sequence $\langle e_1, \ldots, e_{k-1}, e_{k+1}, \ldots, e_n \rangle$. Note that if $k \in \text{MR}(\tau)$ then $\tau \odot k \in \text{Runs}(A)$; it follows from condition 2. of the definition of an asynchronous transition system.

**Definition 2.3.** (Hereditary history preserving bisimulation). Let $A_i = (S_i, s_i^0, E_i, \rightarrow_i, L, \lambda_i, I_i)$, for $i \in \{1, 2\}$, be labelled asynchronous transition systems. A relation $B \subseteq \text{Runs}(A_1) \times \text{Runs}(A_2)$ is a **hereditary history preserving (hhp-) bisimulation** relating $A_1$ and $A_2$ if the following conditions are satisfied:

1. $(\varepsilon, \varepsilon) \in B,$

and if $(\tau_1, \tau_2) \in B$ then $\lambda_1(\tau_1) = \lambda_2(\tau_2)$, and:

2. for all $i \in \{1, 2\}$, and $e_i \in E_i$, if $\tau_i, e_i \in \text{Runs}(A_i)$ then there exists $e_{3-i} \in E_{3-i}$, such that $e_{3-i} \cdot e_i \in \text{Runs}(A_{3-i})$, $\lambda_1(e_1) = \lambda_2(e_2)$, and $(\tau_1 \cdot e_1, \tau_2 \cdot e_2) \in B,$

3. $\text{MR}(\tau_1) = \text{MR}(\tau_2),$

4. if $k \in \text{MR}(\tau_1)$ then $(\tau_1 \odot k, \tau_2 \odot k) \in B.$

Two asynchronous transition systems $A_1$ and $A_2$ are hereditary history preserving (hhp-) **bisimilar**, if there is an hhp-bisimulation relating them. For alternative and slightly varying definitions of hhp-bisimulation that all give rise to the same notion of hhp-bisimilarity see, e.g., the papers by Bednarczyk [1], Joyal et al. [15], Nielsen and Clausen [22], Nielsen and Winskel [23], or Fröschle and Hildebrandt [7].

The following proposition is straightforward since every asynchronous transition system $A$ and its unfolding $\Unf(A)$ have the same set of runs and the same independence relation.

**Proposition 2.2.** Asynchronous transition systems $A_1$ and $A_2$ are hhp-bisimilar if and only if their unfoldings $\Unf(A_1)$ and $\Unf(A_2)$ are hhp-bisimilar.

The main results of this paper are the following theorems proved in Section 4.

**Theorem 2.1** (Undecidability of hhp-bisimilarity).

Hhp-bisimilarity is undecidable for finite labelled asynchronous transition systems.

**Theorem 2.2** (Undecidability of unlabelled hhp-bisimilarity).

Hhp-bisimilarity is undecidable for finite unlabelled asynchronous transition systems.
The process of checking hhp-bisimilarity of asynchronous transition systems is conveniently viewed as a game played on runs of the systems by two players: *Spoiler* and *Duplicator*. Duplicator aims to prove the systems to be bisimilar while Spoiler tries to show the opposite [27, 28, 22].

**Definition 2.4. (Hhp-bisimilarity checking game).**
Let $A_i = (S_i, s_i^{\text{init}}, E_i, \rightarrow_i, L, \lambda_i, I_i)$ for $i \in \{1, 2\}$ be labelled asynchronous transition systems. Configurations of the hhp-bisimilarity checking game $B_{\text{hp}}(A_1, A_2)$ are elements of the set $\text{Runs}(A_1) \times \text{Runs}(A_2)$. Game $B_{\text{hp}}(A_1, A_2)$ is played by two players: *Spoiler* and *Duplicator*. The initial configuration is the pair of empty runs $(\varepsilon, \varepsilon)$. In each move the players change the current configuration $(\tau_1, \tau_2)$ of $B_{\text{hp}}(A_1, A_2)$ in one of the following ways chosen by Spoiler.

- **Forward move:**
  
  1. Spoiler chooses an $i \in \{1, 2\}$ and an event $e_i \in E_i$, such that $\tau_i \cdot e_i \in \text{Runs}(A_i)$;
  
  2. Duplicator responds by choosing an event $e_{3-i} \in E_{3-i}$, such that $\overline{\tau_{3-i}} e_{3-i} \in \text{Runs}(A_{3-i})$, and $\lambda_1(e_1) = \lambda_2(e_2)$;

  the pair $(\tau_1 \cdot e_1, \tau_2 \cdot e_2)$ becomes the current configuration.

- **Backward move:**
  
  1. Spoiler chooses an $i \in \{1, 2\}$ and a $k \in \text{MR}(\tau_i)$;
  
  2. Duplicator can only respond if $k \in \text{MR}(\tau_{3-i})$; otherwise Duplicator gets stuck;

if $k \in \text{MR}(\tau_1)$, and $k \in \text{MR}(\tau_2)$ then $(\tau_1 \ominus k, \tau_2 \ominus k)$ becomes the current configuration.

A *play of $B_{\text{hp}}(A_1, A_2)$* if a *maximal* sequence of configurations formed by players making moves in the fashion described above. Duplicator is the winner in every infinite play; a finite play is lost by the player who is stuck. Note that Spoiler gets stuck only if both transition systems have no transitions going out from their initial states.

We skip the tedious details of formalizing notions of strategies and winning strategies for either of the players. The following standard fact is proved by arguing that an hhp-bisimulation is a good formalization of the notion of a winning strategy for Duplicator in an hhp-bisimilarity checking game [22].

**Proposition 2.3.** *Asynchronous transition systems $A_1$ and $A_2$ are hhp-bisimilar if and only if there is a winning strategy for Duplicator in hhp-bisimilarity checking game $B_{\text{hp}}(A_1, A_2)$.*

It is easy to see how an hhp-bisimulation $B \subseteq \text{Runs}(A_1) \times \text{Runs}(A_2)$ can serve as a winning strategy for Duplicator in $B_{\text{hp}}(A_1, A_2)$. Intuitively, the hhp-bisimulation $B$ contains all configurations of $B_{\text{hp}}(A_1, A_2)$ which can become current configurations when Duplicator is following the strategy determined by $B$. The strategy is defined as follows. Let $(\tau_1, \tau_2)$ be the current configuration of
$B_{hhp}(A_1, A_2)$. If Spoiler chooses event $e_i$ of $A_i$ in a forward move, then Duplicator
respects by choosing an event $e_{A_i}$ of $A_{A_i}$, such that $(\pi_1 \cdot e_1, \pi_2 \cdot e_2) \in B$. If
Spoiler makes a backward move then response of Duplicator is unique. This strategy
contains the initial configuration $(\varepsilon, \varepsilon)$ by condition 1. of the definition of an
$hhp$-bimimulation, and it is well defined by conditions 2.−4.

It is not very hard to argue that bisimilarity checking games are determined.

**Proposition 2.4** (Determinacy). **Bisimilarity checking games are determined,**
i.e., in every game exactly one of the players has a winning strategy.

It follows that if one of the players does not have a winning strategy in a
bisimilarity checking game then the other player does.

### 3. Domino bisimilarity

In this section we introduce a novel algorithmic problem of checking domino
bisimilarity for labelled origin constrained tiling systems and an equivalent
problem of solving domino bisimilarity checking games. The main results are the un-
decidability of the problem and its extension to the unlabelled case. These results
serve crucially in Section 4 to establish the main result of the paper, i.e., the unde-
cidability of $hhp$-bisimilarity for finite, both labelled and unlabelled, asynchronous
transition systems and elementary net systems.

In Subsection 3.1 we define the algorithmic problem of checking domino bisim-
ilarity. In Subsection 3.2 we recall 2-counter machines [20] and in Subsection 3.3
we prove the undecidability of labelled domino bisimilarity by a reduction from the
halting problem for 2-counter machines. Finally, in Subsection 3.4 we extend the
undecidability result to the unlabelled case.

It is worthwhile to compare our domino bisimilarity, or equivalently domino
bisimilarity checking games, to domino snakes studied by Etzion-Petruschka et
al. [5] and domino games of Grädel [10]. Despite apparent similarities of our domino
bisimilarity checking games to the latter, our games seem to be of quite a different
flavour and indeed the proofs of undecidability are very different.

#### 3.1. Domino bisimilarity

**Definition 3.1.** (Origin constrained tiling system).
An **origin constrained tiling system** $T = (D, D^{\text{ori}}, (H, H^0), (V, V^0), L, \lambda)$ consists of
a set $D$ of dominoes, its subset $D^{\text{ori}} \subseteq D$ called the **origin constraint**, two **horizontal**
compatibility relations $H, H^0 \subseteq D^2$, two **vertical** compatibility relations $V, V^0 \subseteq D^2$,
a set $L$ of labels, and a **labelling function** $\lambda: D \to L$.

A **configuration** of $T$ is a triple $(d, x, y) \in D \times \mathbb{N} \times \mathbb{N}$, such that if $x = y = 0$ then
d $\in D^{\text{ori}}$. In other words, in the “origin” position $(x, y) = (0, 0)$ of the non-negative
integer grid only dominoes from the origin constraint $D^{\text{ori}}$ are allowed.

Let $(d, x, y)$, and $(d', x', y')$ be configurations of $T$ such that $|x' - x| + |y' - y| = 1$,
i.e., the positions $(x, y)$, and $(x', y')$ are neighbours in the non-negative integer
grid. Without loss of generality we may assume that $x + y < x' + y'$. We say that
configurations $(d, x, y)$, and $(d', x', y')$ are **compatible** if either of the two conditions
below holds:
\( x' = x \), and \( y' = y + 1 \), and

if \( y = 0 \), then \( (d, d') \in V^0 \), and if \( y > 0 \), then \( (d, d') \in V \), or

- \( x' = x + 1 \), and \( y' = y \), and

if \( x = 0 \), then \( (d, d') \in H^0 \), and if \( x > 0 \), then \( (d, d') \in H \).

**Definition 3.2.** (Domino bisimulation).

Let \( T_i = (D_i, D^0_i, (H_i, H^0_i), (V_i, V^0_i), L_i, \lambda_i) \) for \( i \in \{1, 2\} \) be origin constrained tiling systems. A relation \( B \subseteq D_1 \times D_2 \times \mathbb{N} \times \mathbb{N} \) is a *domino bisimulation* relating \( T_1 \) and \( T_2 \), if \((d_1, d_2, x, y) \in B\) implies that \( \lambda_1(d_1) = \lambda_2(d_2) \), and the following conditions are satisfied for all \( i \in \{1, 2\} \):

1. for all \( d_i \in D^0_i \), there is \( d_{3-i} \in D^0_{3-i} \), such that \( \lambda_1(d_1) = \lambda_2(d_2) \), and \((d_1, d_2, 0, 0) \in B\).

2. for all \( x, y \in \mathbb{N} \), such that \((x, y) \neq (0, 0)\), and \( d_i \in D_i \), there is \( d_{3-i} \in D_{3-i} \), such that \( \lambda_1(d_1) = \lambda_2(d_2) \), and \((d_1, d_2, x, y) \in B\).

3. if \((d_1, d_2, x, y) \in B\), then for all neighbours \((x', y') \in \mathbb{N} \times \mathbb{N}\) of \((x, y)\), and \( d'_i \in D_i \), if configurations \((d_i, x, y)\), and \((d'_i, x', y')\) of \( T_i \) are compatible, then there exists \( d'_{3-i} \in D_{3-i} \), such that \( \lambda_1(d'_1) = \lambda_2(d'_2) \), and configurations \((d_{3-i}, x, y)\), and \((d'_{3-i}, x', y')\) of \( T_{3-i} \) are compatible, and \((d'_1, d'_2, x', y') \in B\).

We say that two tiling systems are *domino bisimilar* if and only if there is a domino bisimulation relating them.

The main result of this section is the following theorem proved in Subsection 3.3.

**Theorem 3.1.** (Undecidability of domino bisimilarity).

Domino bisimilarity is undecidable for origin constrained tiling systems.

The proof is a reduction from the halting problem for deterministic 2-counter machines. For a deterministic 2-counter machine \( M \) we define in Subsection 3.3 two origin constrained tiling systems \( T_1 \), and \( T_2 \), such that the following holds.

**Lemma 3.1.** Machine \( M \) does not halt if and only if there is a domino bisimulation relating \( T_1 \) and \( T_2 \).

The process of checking domino bisimilarity of origin constrained tiling systems is conveniently viewed as a game played on an infinite grid by two players: Spoiler and Duplicator. As in the case of hhp-bisimilarity checking games Duplicator aims to prove the tiling systems to be bisimilar while Spoiler tries to show the opposite.

**Definition 3.3.** (Origin constrained domino bisimilarity checking game).

Let \( T_1 \) and \( T_2 \) be origin constrained tiling systems. *Configurations of the origin constrained domino bisimilarity checking game \( B_3(T_1, T_2) \) are elements of the set \( D_1 \times D_2 \times \mathbb{N} \times \mathbb{N} \). Game \( B_3(T_1, T_2) \) is played by two players Spoiler and Duplicator.*

- First the players fix an initial configuration:

1. Spoiler chooses an \( i \in \{1, 2\} \), and a configuration \((d_i, x, y)\) of \( T_i \),

2. Duplicator responds by choosing a domino \( d_{3-i} \in D_{3-i} \), such that \((d_{3-i}, x, y)\)

is a configuration of \( T_{3-i} \), and \( \lambda_1(d_1) = \lambda_2(d_2) \);
if both players were able to make their choices then the tuple \((d_1, d_2, x, y)\) becomes the current configuration of \(B_d(T_1, T_2)\).

- In each move of the domino bisimilarity checking game the players change the current configuration \((d_1, d_2, x, y)\):

1. Spoiler chooses an \(i \in \{1, 2\}\), and a configuration \((d'_i, x', y')\) of \(T_i\) compatible with configuration \((d_i, x, y)\),

2. Duplicator responds by choosing a domino \(d'_{3-i} \in D_{3-i}\) so that \((d'_{3-i}, x', y')\) is a configuration of \(T_{3-i}\), and \(\lambda_1(d_1) = \lambda_2(d_2)\), and configurations \((d_{3-i}, x, y)\) and \((d'_{3-i}, x', y')\) of \(T_{3-i}\) are compatible;

if both players were able to make their choices then the tuple \((d'_1, d'_2, x', y')\) becomes the current configuration of \(B_d(T_1, T_2)\).

A play of \(B_d(T_1, T_2)\) is a maximal sequence of configurations formed by players making moves in the fashion described above. Duplicator is the winner in every infinite play; a finite play is lost by the player who is stuck.

We avoid tedious details of formalizing notions of strategies and winning strategies for either of the players. The following simple fact is proved by arguing that a domino bisimulation is a good formalization of a winning strategy for Duplicator in a domino bisimilarity checking game.

Proposition 3.1. Origin constrained tiling systems \(T_1\) and \(T_2\) are domino bisimilar if and only if Duplicator has a winning strategy in the domino bisimilarity game \(B_d(T_1, T_2)\).

As in the case of hh-p-bisimilarity checking games it is easy to argue that domino bisimilarity checking games are determined; in other words Proposition 2.4 holds also for domino bisimilarity checking games.

### 3.2. Counter machines

A 2-counter machine \(M\) consists of a finite program with the set \(L\) of instruction labels, and instructions of the form:

- \(\ell: \ c_i := c_i + 1; \ \text{goto} \ m\)
- \(\ell: \ \text{if} \ c_i = 0 \ \text{then} \ c_i := c_i + 1; \ \text{goto} \ m\)
- \(\text{else} \ c_i := c_i - 1; \ \text{goto} \ n\)
- \(\text{halt}\)

where \(i = 1, 2; \ \ell, m, n \in L, \text{and } \{\text{start, halt}\} \subseteq L\). A configuration of \(M\) is a triple \((\ell, x, y)\) ∈ \(L × N × N\), where \(\ell\) is the label of the current instruction, and \(x\) and \(y\) are the values stored in counters \(c_1\) and \(c_2\), respectively; we denote the set of configurations of \(M\) by \(\text{Conf}_M\). The semantics of 2-counter machines is standard: let \(\vdash_M \subseteq \text{Conf}_M \times \text{Conf}_M\) be the usual one-step derivation relation on configurations of \(M\); by \(\vdash^*_M\) we denote the reachability (in at least one step) relation for configurations, i.e., the transitive closure of \(\vdash_M\).
Before we give a reduction from the halting problem of 2-counter machines to origin constrained domino bisimilarity let us take a look at the directed graph \((\text{Confs}(M), \vdash_M)\), with configurations of \(M\) as vertices, and edges denoting derivation in one step. Since machine \(M\) is deterministic, for each configuration there is at most one outgoing edge; moreover only halting configurations have no outgoing edges. It follows that connected components of the graph \((\text{Confs}(M), \vdash_M)\) are either trees with edges going to the root which is the unique halting configuration in the component, or have no halting configuration at all. This observation is formalized in the following proposition.

**Proposition 3.2.** Let \(M\) be a 2-counter machine. The following conditions are equivalent:

1. machine \(M\) halts on input \((0,0)\), i.e., \((\text{start},0,0) \vdash_M^+ (\text{halt},x,y)\) for some \(x, y \in \mathbb{N}\),
2. \((\text{start},0,0) \sim_M (\text{halt},x,y)\) for some \(x, y \in \mathbb{N}\), where the relation \(\sim_M \subseteq \text{Confs}(M) \times \text{Confs}(M)\) is the symmetric and transitive closure of \(\vdash_M\).

### 3.3. The reduction

In this subsection we give a proof of Theorem 3.1 by proving Lemma 3.1. The idea is to design a tiling system which “simulates” the behaviour of a 2-counter machine.

Let \(M\) be a 2-counter machine. We construct a tiling system \(T_M\) with the set \(L\) of instruction labels of \(M\) as the set of dominoes, and the identity function on \(L\) as the labelling function. Note that this implies that all tuples belonging to a domino bisimulation relating copies of \(T_M\) are of the form \((\ell, \ell, x, y)\), so we can identify them with configurations of \(M\), i.e., sometimes we will make no distinction between \((\ell, \ell, x, y)\) and \((\ell, x, y) \in \text{Confs}(M)\) for \(\ell \in L\).

We define the horizontal compatibility relations \(H_M, H_M^0 \subseteq L \times L\) of the tiling system \(T_M\) as follows:

- \((\ell, m) \in H_M\) if and only if either of the instructions below is an instruction of machine \(M\):
  - \(\ell:\ c_1 := c_1 + 1; \text{ goto } m\)
  - \(m: \text{ if } c_1 = 0 \text{ then } c_1 := c_1 + 1; \text{ goto } n \)
  - \(\text{ else } c_1 := c_1 - 1; \text{ goto } \ell\)

- \((\ell, m) \in H_M^0\) if and only if \((\ell, m) \in H_M\), or the instruction below is an instruction of machine \(M\):
  - \(\ell: \text{ if } c_1 = 0 \text{ then } c_1 := c_1 + 1; \text{ goto } m \)
  - \(\text{ else } c_1 := c_1 - 1; \text{ goto } n\)

Vertical compatibility relations \(V_M\) and \(V_M^0\) are defined in the same way, with \(c_1\) instructions replaced with \(c_2\) instructions. We also take \(D_M^0 = L\), i.e., all dominoes are allowed in position \((0,0)\). Note that the identity function is a 1-1 correspondence between configurations of \(M\), and configurations of the tiling system \(T_M\); from now
on we will hence identify configurations of $M$ and $T_M$. It follows immediately from 
the construction of $T_M$, that two configurations $c, c' \in \text{Conf}(M)$ are compatible as 
configurations of $T_M$, if and only if $c \vdash_M c'$, or $c' \vdash_M c$, i.e., compatibility relation of 
$T_M$ coincides with the symmetric closure of $\vdash_M$. By $\approx_M$ we denote the symmetric 
and transitive closure of the compatibility relation of configurations of $T_M$. The 
following proposition is then straightforward.

**Proposition 3.3.** The two relations $\sim_M$ and $\approx_M$ coincide.

Now we are ready to define the two origin constrained tiling systems $T_1$, and $T_2$, 
postulated in Lemma 3.1. The idea is to have two independent and slightly pruned copies of $T_M$ in $T_2$: one without the initial configuration $(\text{start}, 0, 0)$, and the other 
without any halting configurations $(\text{halt}, x, y)$. The other tiling system $T_1$ is going 
to have three independent copies of $T_M$: the two of $T_2$, and moreover, another full 
copy of $T_M$.

More formally we define $D_2 = (L \times \{1, 2\}) \setminus \{(\text{halt}, 2)\}$, and $D_2^{ri} = D_2 \setminus 
\{(\text{start}, 1)\}$, and $V_2 = \left((V_M \oplus 1) \cup (V_M \oplus 2)\right) \cap (D_2 \times D_2)$, where for a binary 
relation $R$ we define $R \otimes i$ to be the relation $\{(a, i), (b, i)\} : (a, b) \in R$. The other 
compatibility relations $V_2^0$, $H_2$, and $H_2^0$ are defined analogously from the respective 
compatibility relations of $T_M$.

The tiling system $T_1$ is obtained from $T_2$ by adding yet another independent copy 
of $T_M$, this time a complete one: $D_1 = D_3 \cup (L \times \{3\})$, and $D_1^{ri} = D_2^{ri} \cup (L \times \{3\})$, 
and $V_1 = V_2 \cup (V_M \oplus 3)$, etc. The labelling functions of $T_1$, and $T_2$ are defined as 
$\lambda_1((\ell, i)) = \ell$.

In order to show Lemma 3.1, and hence conclude the proof of Theorem 3.1, it 
suffices to establish the following two lemmas.

**Lemma 3.2.** If machine $M$ halts on input $(0, 0)$ then origin constrained tiling 
systems $T_1$ and $T_2$ are not domino bisimilar.

**Proof.** By Proposition 3.1 it suffices to show that if machine $M$ halts on input 
$(0, 0)$ then Spoiler has a winning strategy in the game $B_d(T_1, T_2)$. Spoiler 
starts by choosing the configuration $((\text{start}, 3), 0, 0)$ of $T_1$. Duplicator has to 
respond with domino $(\text{start}, 2)$ of $T_2$ since $(\text{start}, 1) \notin D_2^{ri}$. Then Spoiler 
“simulates” the finite computation of $M$ on input $(0, 0)$ in the following way. If 
$((\ell, 3), (\ell, 2), x, y)$ is the current configuration of the game then Spoiler chooses the 
configuration $((\ell', 3), x', y')$ of $T_1$, such that $(\ell, x, y) \vdash_M (\ell', x', y')$. This 
move is allowed thanks to Proposition 3.3. Then Duplicator can only respond 
with domino $(\ell', 2)$ of $T_2$, and $((\ell', 3), (\ell', 2), x', y')$ becomes the current configuration 
of the game. In the last step of the simulation Spoiler chooses a configuration 
$((\text{halt}, 3), x', y')$ for some $x', y' \in \mathbb{N}$ which makes Duplicator stuck because 
$(\text{halt}, 2) \notin D_2$. $
$
**Lemma 3.3.** If machine $M$ does not halt on input $(0, 0)$ then origin constrained 
tiling systems $T_1$ and $T_2$ are domino bisimilar.
Proof. By Proposition 3.1 it suffices to show that if machine \( M \) does not halt on input \((0, 0)\) then Duplicator has a winning strategy in the game \( B_M(T_1, T_2) \). We claim that the following is a winning strategy for Duplicator:

If in the first step Spoiler chooses a configuration \(((\ell, j), x, y)\) of \( T_1 \) or \( T_2 \) for \( j \in \{1, 2\} \), then Duplicator responds with the domino \((\ell, j)\) of the other tiling system. It is obvious that then Duplicator can respond to all moves of Spoiler because both players play on identical pruned copies of \( T_M \).

If instead Duplicator chooses a configuration \(((\ell, 3), x, y)\) of \( T_1 \) in the first step then Duplicator responds with:

- domino \((\ell, 1)\) of \( T_2 \) if \((\ell, x, y) \sim_M (\text{halt}, x', y')\) or \((\ell, x, y) = (\text{halt}, x', y')\) for some \( x', y' \in \mathbb{N} \), and
- domino \(((\ell, 2), x, y) \neq_M (\text{halt}, x', y')\) for all \( x', y' \in \mathbb{N} \).

In the first case the only way Spoiler can make Duplicator stuck is to be able to choose configuration \(((\text{start}, 3), 0, 0)\) of \( T_1 \) since the only difference between copy 3 of \( T_M \) in \( T_1 \) and copy 1 of \( T_M \) in \( T_2 \) is that the latter does not have the triple \(((\text{start}, 0, 0)\) as a configuration. Hence in order to prove that Duplicator has a winning strategy from the initial configuration \(((\ell, 3), (\ell, 1), x, y)\), it suffices to show that \((\ell, x, y) \neq_M (\text{start}, 0, 0)\). Assume for the sake of contradiction that \((\ell, x, y) \neq_M (\text{start}, 0, 0)\). By Proposition 3.3 we then have \((\ell, x, y) \sim_M (\text{start}, 0, 0)\). This, by our assumption that \((\ell, x, y) \sim_M (\text{halt}, x', y')\) for some \( x', y' \in \mathbb{N} \), implies that \((\text{start}, 0, 0) \sim_M (\text{halt}, x', y')\) for some \( x', y' \in \mathbb{N} \). Then Proposition 3.2 implies that \((\text{start}, 0, 0) \vdash_M (\text{halt}, x', y')\), which contradicts the assumption of the lemma that machine \( M \) does not halt on input \((0, 0)\).

The argument in the other case is similar. It suffices to show that \((\ell, x, y) \neq_M (\text{halt}, x', y')\) for all \( x', y' \in \mathbb{N} \), because the only difference between copy 3 of \( T_M \) in \( T_1 \) and copy 2 of \( T_M \) in \( T_2 \) is that the latter has no triple \((\text{halt}, x', y')\) as a configuration. By applying Proposition 3.3 to our assumption that \((\ell, x, y) \neq_M (\text{halt}, x', y')\) for all \( x', y' \in \mathbb{N} \), we immediately get that \((\ell, x, y) \neq_M (\text{halt}, x', y')\) for all \( x', y' \in \mathbb{N} \).

3.4. Undecidability of unlabelled domino bisimilarity

In this section we argue that the problem of deciding domino bisimilarity is undecidable even for unlabelled origin constrained tiling systems or, equivalently, for origin constrained tiling systems with a singleton set of labels.

Theorem 3.2 (Undecidability of unlabelled domino bisimilarity).

Unlabelled domino bisimilarity is undecidable for origin constraint tiling systems.

Proof. We show that the unlabelled origin constrained domino bisimilarity checking game is undecidable and then we use Proposition 3.1.

Let \( T_i = (D_i, D_i^0, (H_i, H_i^0), (V_i, V_i^0), L_i, \lambda_i) \), for \( i \in \{1, 2\} \), be origin constrained tiling systems. Without loss of generality we can assume that \( L_1 = L_2 = \{1, \ldots, n\} \), for some \( n \geq 1 \). We give an effective construction of unlabelled origin constrained tiling systems \( T_i' = (D_i', D_i'^0, (H_i', H_i'^0), (V_i', V_i'^0)), \) for \( i \in \{1, 2\} \), such that the following holds.
Lemma 3.4 (The reduction).

Duplicator has a winning strategy in the labelled domino bisimilarity checking game $B_d(T_1, T_2)$ if and only if he has a winning strategy in the unlabelled game $B_d(T_1, T_2)$.

Establishing this lemma will complete the proof of Theorem 3.2. Tiling systems $\overline{T}_i$, for $i \in \{1, 2\}$, are defined as follows:

- $\overline{D}_i = D_i \cup \{ c^i_d, \ldots, c^{\lambda_i(d)}_d : d \in D_i \} \cup \{ b_d : d \in D_i \}$, the dominoes in $D_i$ are called \textit{plain dominoes} and the ones in $\overline{D}_i \setminus D_i$ are called \textit{auxiliary dominoes},

- $\overline{H}_i = H_i \cup \{ (d, c^i_d) : d \in D_i \} \cup \{ (c^i_d, c^{i+1}_d) : d \in D_i \text{ and } 1 \leq i < \lambda_i(d) \}$,

- $\overline{H}_i^0 = H_i^0 \cup \{ (d, c^i_d) : d \in D_i \} \cup \{ (c^i_d, c^{i+1}_d) : d \in D_i \text{ and } 1 \leq i < \lambda_i(d) \}$,

- $\overline{V}_i = V_i \cup \{ (d, b_d) : d \in D_i \}$,

- $\overline{V}_i^0 = V_i^0 \cup \{ (d, b_d) : d \in D_i \}$.

For the proof of the “if” part of Lemma 3.4 the following two facts will be instrumental.

Lemma 3.5. Let $(d_1, d_2, x, y)$ be a configuration of the unlabelled domino bisimulation checking game $B_d(T_1, T_2)$, such that $d_i$ is a plain domino and $d_{3-i}$ is an auxiliary domino, for some $i \in \{1, 2\}$. Then Spoiler has a winning strategy from this configuration in the unlabelled game $B_d(T_1, T_2)$.

Proof. Consider the case $i = 1$; the other case is symmetric. Spoiler wins immediately by making the move $(b_{d_1}, x, y + 1)$: Duplicator has no response to this move. \hfill \blacksquare

Lemma 3.6. Let $(d_1, d_2, x, y)$ be a configuration of the unlabelled domino bisimilarity checking game $B_d(T_1, T_2)$, such that $\lambda_1(d_1) \neq \lambda_2(d_2)$. Then Spoiler has a winning strategy from this configuration in the unlabelled game $B_d(T_1, T_2)$.

Proof. Without loss of generality assume that $\lambda_1(d_1) > \lambda_2(d_2)$. Let Spoiler play the following sequence of moves from the configuration $(d_1, d_2, x, y)$ in the unlabelled game $B_d(T_1, T_2)$:

$$(c^{\lambda_1(d_2)}_{d_1}, x + 1, y), (c^{\lambda_2(d_2)}_{d_2}, x + 2, y), \ldots, (c^{\lambda_2(d_2)}_{d_2}, x + \lambda_2(d_2), y).$$

The only way for Duplicator to avoid losing immediately as in Lemma 3.5 is to respond with the following sequence of moves:

$$(c^{\lambda_1(d_2)}_{d_1}, x + 1, y), (c^{\lambda_2(d_2)}_{d_2}, x + 2, y), \ldots, (c^{\lambda_2(d_2)}_{d_2}, x + \lambda_2(d_2), y).$$

From the configuration $(c^{\lambda(d_2)}_{d_1}, c^{\lambda_2(d_2)}_{d_2}, x + \lambda_2(d_2), y)$ Spoiler wins immediately by playing the move $(c^{\lambda(d_2) + 1}_{d_1}, x + \lambda_2(d_2) + 1, y)$: Duplicator has no response to this move. \hfill \blacksquare
We are now ready to establish the “if” part of Lemma 3.4.

**Lemma 3.7 ("if").**

If Duplicator has a winning strategy in the unlabelled domino bisimilarity checking game $B_d(T_1, T_2)$ then he has a winning strategy in the labelled game $B_d(T_1, T_2)$.

**Proof.** Note that every configuration of the labelled domino bisimilarity checking game $B_d(T_1, T_2)$ is also a configuration of the unlabelled domino bisimilarity checking game $B_d(T_1, T_2)$. Given a winning strategy for Duplicator in the unlabelled game $B_d(T_1, T_2)$ we define a strategy for Duplicator in the labelled game $B_d(T_1, T_2)$ to be equal to the restriction of the former strategy to configurations and moves of the labelled game $B_d(T_1, T_2)$. This strategy is well defined because of Lemmas 3.5 and 3.6; it is clearly a winning strategy.

Finally, we conclude the proof of Theorem 3.2 by sketching a proof of the “only if” part of Lemma 3.4.

**Lemma 3.8 ("only if").**

If Duplicator has a winning strategy in the labelled domino bisimilarity checking game $B_d(T_1, T_2)$ then he has a winning strategy in the unlabelled game $B_d(T_1, T_2)$.

**Proof.** Suppose that Duplicator has a winning strategy in the labelled domino bisimilarity checking game $B_d(T_1, T_2)$. A configuration of the unlabelled game $B_d(T_1, T_2)$ is called *admissible* if it is also a configuration of the labelled game $B_d(T_1, T_2)$ and it belongs to the winning strategy for Duplicator there, i.e., if it is reachable by Duplicator playing the strategy. We define the strategy for Duplicator in the unlabelled game $B_d(T_1, T_2)$ to be equal to the strategy in the labelled game for all admissible configurations and moves in which Spoiler chooses a plain domino. We need to define the strategy for Duplicator for the configurations that are not admissible or moves in which Spoiler chooses an auxiliary domino.

The strategy we define for Duplicator in the unlabelled game $B_d(T_1, T_2)$ has the property that every configuration $C = (d_1, d_2, x, y)$ that belongs to the strategy, i.e., that can be reached when Duplicator follows the strategy, is one of the following forms:

1. The configuration $C$ belongs to the winning strategy for Duplicator in the labelled game $B_d(T_1, T_2)$:
   2. $d_1 = c_{d_3}^k$ and $d_2 = c_{d_4}^k$, for some plain dominoes $d_3 \in D_1$ and $d_4 \in D_2$, and $\lambda_1(d_3) = \lambda_2(d_4)$, and if $x \geq k$ then the configuration $(d_3, d_4, x - k, y)$ belongs to the winning strategy for Duplicator in the labelled game $B_d(T_1, T_2)$;
   3. $d_1 = b_{d_3}$ and $d_2 = b_{d_4}$, for some plain dominoes $d_3 \in D_1$ and $d_4 \in D_2$, and $\lambda_1(d_3) = \lambda_2(d_4)$, and if $y > 0$ then the configuration $(d_3, d_4, x, y - 1)$ belongs to the winning strategy for Duplicator in the labelled game $B_d(T_1, T_2)$.

It is a tedious but routine exercise to inspect that if Spoiler plays from a configuration of one of the forms 1–3. listed above then Duplicator can always respond so as to make the next configuration fall into one of the categories 1–3. ■
4. HHP-BISIMILARITY IS UNDECIDABLE

The main result of this section is a proof of Theorem 2.1, i.e., the undecidability of hhp-bisimilarity for finite state asynchronous transition systems. We also give a few extensions of this result: undecidability of hhp-bisimilarity for finite unlabelled asynchronous transition systems and for finite (unlabelled) elementary net systems.

The proof of our main result is a reduction from the problem of deciding domino bisimilarity for origin constrained tiling systems shown to be undecidable in the previous section. The method of encoding a tiling system on an infinite grid in the unfolding of a finite asynchronous transition system is due to Madhusudan and Thiagarajan [18]. For each origin constrained tiling system $T$, we give an effective definition a finite asynchronous transition system $A(T)$, such that the following holds.

**Lemma 4.1** (The reduction). Origin constrained tiling systems $T_1$ and $T_2$ are domino bisimilar if and only if asynchronous transition systems $A(T_1)$ and $A(T_2)$ are hhp-bisimilar.

In Subsections 4.1–4.3 we define the finite asynchronous transition system $A(T)$ and prove Lemma 4.1 thus completing the proof of Theorem 2.1. In Subsection 4.4 we prove Theorem 2.2, i.e., we show that hhp-bisimilarity is undecidable even for unlabelled finite asynchronous transition systems. Finally, in Subsection 4.5 we argue that the hhp-bisimilarity undecidability results hold for the class of asynchronous transition systems induced by elementary net systems.

4.1. Asynchronous transition system $A(T)$

Let $T = (D, D^o, (H, H^0), (V, V^0), L, \lambda)$ be an origin constrained tiling system. The infinite grid structure is modelled by the unfolding of the asynchronous transition system shown in Figure 1. The set of events of this asynchronous transition
system is \( E = \{ x_0, x_1, x_2, x_3, y_0, y_1, y_2, y_3, y_4 \} \). The independence relation \( I \) is the symmetric closure of \( \{ (x_i, y_j) : i, j \in \{0, 1, 2, 3, 4\} \} \).

We identify the states of the asynchronous transition system in Figure 1 with pairs of numbers \((i, j) \in \{0, 1, 2, 3, 4\}^2\), where \(i\) is the horizontal coordinate and \(j\) is the vertical coordinate. The state in the bottom-left corner in Figure 1 is \((0, 0)\); it is the initial state. For all \(n \in \mathbb{N}\), define:

\[
\tilde{n} = \begin{cases} 
  n & \text{if } n \leq 4, \\
  2 + ((n - 2) \mod 3) & \text{if } n > 4.
\end{cases}
\]

A position \((n, m) \in \mathbb{N}^2\) of the infinite grid is represented by state \((\tilde{n}, \tilde{m})\) in the asynchronous transition system \(A(T)\).

Configurations of the tiling system \(T\) are modelled by extra transitions going out of states of the grid structure in Figure 1, and labelled by events of the form \(d_{ij}\), for \(d \in D\), and \(i, j \in \{0, 1, 2, 3\}\). We define a set of events \(E_D\) as follows:

\[
E_D = \{ (d_{ij}) : d \in D; \text{ and } i, j \in \{0, 1, 2, 3\}; \text{ and } i = j = 0 \implies d \in D^{\text{ext}} \}.
\]

The idea is, for every \(d \in D\), to have a transition going out of each state \((i, j) \in \{0, 1, 2, 3, 4\}^2\) labelled with \(d_{ij}\), provided that \((d, i, j)\) is a configuration of \(T\). In fact, for a technical reason we need to use events \(d_{i1}\) and \(d_{ij}\) at states \((i, 4)\) and \((4, j)\), for \(i, j \in \{0, 1, 2, 3\}\), respectively, instead of \(d_{i4}\) and \(d_{4j}\). In order to avoid special treatment of this case throughout the rest of the paper we adopt the following notation, for all \(n \in \mathbb{N}\):

\[
\tilde{n} = \begin{cases} 
  n & \text{if } n \leq 3, \\
  1 + ((n - 1) \mod 3) & \text{if } n > 3.
\end{cases}
\]

Horizontal and vertical compatibility relations for configurations of the tiling system \(T\) are modelled by an independence relation \(I_D\) on \(E_D\), according to which events \(d_{ij}\) and \(e_{k}\) corresponding to “neighbouring” configurations are independent if and only if the configurations are compatible. More precisely, we define \(I_D\) to be the symmetric closure of the following set:

\[
\{ (d_{ij}, e_{(i+1)j}) : j \in \{0, 1, 2, 3\} \text{ and } (d, e) \in H_0 \} \cup \\
\{ (d_{ij}, e_{i(j+1)}) : i \in \{1, 2, 3\}, j \in \{0, 1, 2, 3\}, \text{ and } (d, e) \in H \} \cup \\
\{ (d_{i0}, e_{i1}) : i \in \{0, 1, 2, 3\} \text{ and } (d, e) \in V_0 \} \cup \\
\{ (d_{ij}, e_{ij(j+1)}) : i \in \{0, 1, 2, 3\}, j \in \{1, 2, 3\}, \text{ and } (d, e) \in V \}.
\]

For all \(i, j \in \{0, 1, 2, 3, 4\}\), and \(d \in D\), we have up to four transitions going out of state \((i, j)\) and labelled by the following events in \(E_D\): \(d_{i+1,j}^{+}, d_{i-1,j}^{+}\), if \(i > 0\), \(d_{i,j-1}^{-}\) if \(j > 0\), and \(d_{i-1,j-1}^{-}\) if \(i, j > 0\). We write \((i, j, \{d_{ij}\})\) to denote the state reached by the transition labelled by the event \(d_{ij}\) going out of state \((i, j)\).

In other words, for all \(i, j \in \{0, 1, 2, 3, 4\}\) we have the following transitions:

- \((i, j) \xrightarrow{d_{ij}^{+}} (i, j, \{d_{ij}^{+}\})\),

- \((i, j) \xrightarrow{d_{ij}^{-}} (i, j, \{d_{ij}^{-}\})\),

- \((i, j) \xrightarrow{d_{i,j-1}^{-}} (i, j, \{d_{i,j-1}^{-}\})\),

- \((i, j) \xrightarrow{d_{i-1,j}^{-}} (i, j, \{d_{i-1,j}^{-}\})\).
\begin{itemize}
  \item \((i, j, d_{(i-1)j})\) if \(i > 0\),
  \item \((i, j) \xrightarrow{\epsilon} (i, j, \{d_{(i-1)j}\})\) if \(j > 0\),
  \item \((i, j) \xrightarrow{d_{(i-1)j}} (i, j, \{d_{(i-1)(j-1)}\})\) if \(i, j > 0\).
\end{itemize}

Moreover, if there are transitions:

\begin{itemize}
  \item \((i, j) \xrightarrow{d_{k, \ell}} (i, j, \{d_{k, \ell}\})\), and
  \item \((i, j) \xrightarrow{\epsilon} (i, j, \{e_{k, \ell}\})\),
\end{itemize}

and \((d_{k, \ell}, e_{k, \ell'}) \in I_D\), then there is also a state \((i, j, \{d_{k, \ell}, e_{k, \ell'}\})\) and transitions:

\begin{itemize}
  \item \((i, j, \{d_{k, \ell}\}) \xrightarrow{\epsilon} (i, j, \{d_{k, \ell}, e_{k, \ell'}\})\), and
  \item \((i, j, \{e_{k, \ell'}\}) \xrightarrow{d_{k, \ell}} (i, j, \{d_{k, \ell}, e_{k, \ell'}\})\).
\end{itemize}

Finally, there are transitions:

\begin{itemize}
  \item \((i, j, \{d_{2, j'}\}) \xrightarrow{\gamma_j} (i, j+1, \{d_{2, j'}\})\), if \((i, j, \{d_{2, j'}\})\) is a state, and
  \item \((i, j, \{d_{i-1, j}\}) \xrightarrow{\gamma_j} (i, j+1, \{d_{i-1, j}\})\), if \((i, j, \{d_{i-1, j}\})\) is a state,
\end{itemize}

and transitions:

\begin{itemize}
  \item \((i, j, \{d_{2, j'}\}) \xrightarrow{\gamma_j} (i, j+1, \{d_{2, j'}\})\), if \((i, j, \{d_{2, j'}\})\) is a state, and
  \item \((i, j, \{d_{i-1, j}\}) \xrightarrow{\gamma_j} (i, j+1, \{d_{i-1, j}\})\), if \((i, j, \{d_{i-1, j}\})\) is a state.
\end{itemize}

The sets of states \(S_{A(T)}\) and transitions \(\rightarrow_{A(T)}\) of the asynchronous transition system \(A(T) = (S_{A(T)}, \rightarrow_{A(T)}, \lambda_{A(T)}, I_{A(T)})\) are as described above. The set of events is defined by \(E_{A(T)} = E \cup E_D\). The initial state is \(s_{A(T)}^{init} = (0, 0)\).

The independence relation \(I_{A(T)}\) is defined as the symmetric closure of the set:

\[ I \cup I_D \cup \{(x_i, d_{k, \ell}, y_j, d_{k, \ell'}) : i, j \in \{0, 1, 2, 3, 4\} \text{ and } d_{k, \ell}, d_{k, \ell'} \in E_D\}.\]

Finally, the labelling function \(\lambda_{A(T)}\) is an identity on \(E\), and for elements of \(E_D\) it replaces the dominoes with their labels in the tiling system \(T\), i.e.,

\[ \lambda_{A(T)}(e) = \begin{cases} e & \text{if } e \in E, \\ (\lambda(d))_{ij} & \text{if } e \in E_D \text{ and } e = d_{ij}. \end{cases} \]

\textbf{Proposition 4.1.} The labelled transition systems \(A(T)\) is a labelled asynchronous transition system.

\section{The unfolding of \(A(T)\)}

In this subsection we sketch the structure of the unfolding \(\text{Unf}(A(T))\) of asynchronous transition system \(A(T)\) defined in the previous subsection.

For notational convenience we will write \((i, j, 0)\) for a state \((i, j)\) of \(A(T)\). In order to avoid heavy use of notations \(\hat{n}\) and \(\hat{m}\) we adopt the following conventions:

\begin{itemize}
  \item we write \(x_n\) and \(y_m\), for all \(n, m \in \mathbb{N}\), to denote events \(x_n, y_m \in E\), respectively;
\end{itemize}
we write $d_{nm}$ to denote an event $d_{nm} \in E_D$, for all $n,m \in \mathbb{N}$.

**Proposition 4.2.** The set of states of $\text{Unf}(A(T))$ reachable from the initial state $(0, 0, 0)$ consists of triples $(n, m, C) \in \mathbb{N} \times \mathbb{N} \times \wp(E_D)$, such that either:

- $C = \emptyset$; or
- $C = \{d_{nm'}\}$ such that $d_{nm'} \in E_D$, and $n' \in \{n - 1, n\}$, and $m' \in \{m - 1, m\}$; or
- $C = \{d_{(n-1)m'}, \epsilon_{nm'}\}$ such that $d_{(n-1)m'}, \epsilon_{nm'} \in E_D$, and $m' \in \{m - 1, m\}$, and configurations $(d, n - 1, m')$ and $(\epsilon, n, m)$ of $T$ are compatible; or
- $C = \{d_{nm'}(m-1), \epsilon_{nm}\}$ such that $d_{nm'(m-1)}, \epsilon_{nm} \in E_D$, and $n' \in \{n - 1, n\}$, and configurations $(d, n', m - 1)$ and $(\epsilon, n', m)$ of $T$ are compatible.

States of the first category above represent positions on the infinite grid; in particular the state $(n, m, 0)$ represents the position $(n, m) \in \mathbb{N} \times \mathbb{N}$. States of the second category above represent configurations of the tiling system $T$; in particular configuration $(d, n, m) \in D \times \mathbb{N} \times \mathbb{N}$ is represented by states $(n', m', \{d_{nm'}\})$ for $n' \in \{n, n + 1\}$ and $m' \in \{m, m + 1\}$. States of the third and fourth categories above are used to "check compatibility" of neighbouring configurations of the tiling system $T$.

**Proposition 4.3.** The set of transitions of $\text{Unf}(A(T))$ consists of the following:

- $(n, m, C) \xrightarrow{s, \epsilon_{nm}} \text{Unf}(A(T)) (n + 1, m, C)$ for $C = \emptyset$, or for $C = \{d_{nm'}\}$, where $m' \in \{m - 1, m\}$.
- $(n, m, C) \xrightarrow{s, \epsilon_{nm}} \text{Unf}(A(T)) (n, m + 1, C)$ for $C = \emptyset$, or for $C = \{d_{nm}\}$, where $n' \in \{n - 1, n\}$,
- $(n, m, \{d_{(n-1)m'}\}) \xrightarrow{s, \epsilon_{nm'}} \text{Unf}(A(T)) (n, m, \{d_{(n-1)m'}, \epsilon_{nm'}\})$ and
- $(n, m, \{\epsilon_{nm}\}) \xrightarrow{s, d_{nm'}} \text{Unf}(A(T)) (n, m, \{d_{(n-1)m'}, \epsilon_{nm}\})$,

for $n' \in \{n - 1, n\}$, and $m' \in \{m - 1, m\}$, and $d_{nm'} \in E_D$.

- $(n, m, \{d_{nm'(m-1)}\}) \xrightarrow{s, \epsilon_{nm'}} \text{Unf}(A(T)) (n, m, \{d_{nm'(m-1)}, \epsilon_{nm'}\})$ and
- $(n, m, \{\epsilon_{nm}\}) \xrightarrow{s, d_{nm'(m-1)}} \text{Unf}(A(T)) (n, m, \{d_{nm'(m-1)}, \epsilon_{nm}\})$

for $n' \in \{n - 1, n\}$, if configurations $(d, n - 1, m')$ and $(\epsilon, n, m')$ of $T$ are compatible.

### 4.3. Translations between hlp- and domino bisimulations

By Proposition 2.2 it follows that in order to prove Lemma 4.1 it suffices to demonstrate that a domino bisimulation relating $T_1$ and $T_2$ gives rise to an hlp-bisimulation relating $\text{Unf}(A(T_1))$ and $\text{Unf}(A(T_2))$, and vice versa.

In other words, by Propositions 2.3 and 3.1, it suffices to argue that a winning strategy for Duplicator in $B_4(T_1, T_2)$ can be translated to a winning strategy for him in $B_{\text{hlp}}(\text{Unf}(A(T_1)), \text{Unf}(A(T_2)))$, and vice versa. In what follows, in order to keep the arguments from becoming too dull or cumbersome, we are mixing freely
at our convenience the two ways of talking about bisimulations: as relations, and
as winning strategies in bisimilarity checking games.

For notational convenience we introduce the following convention for writing
elements of an hhp-bisimulation relating $\unf(A(T_1))$ and $\unf(A(T_2))$, or equivalently, for configurations of game $B_{\text{hhp}}(\unf(A(T_1)), \unf(A(T_2)))$. Note that if a pair of runs $(\tau_1, \tau_2) \in \text{Runs}(\unf(A(T_1))) \times \text{Runs}(\unf(A(T_2)))$ belongs to an hhp-bisimulation then the states reached by these runs are of the forms $(n, m, C_1)$ and $(n, m, C_2)$ for some $n, m \in \mathbb{N}$, respectively. In what follows we write $(n, m, C_1, C_2)$ to denote such a pair $(\tau_1, \tau_2)$. This notation is a bit sloppy because it is not 1-1.

For example, $(1, 1, \emptyset)$ is used to denote both $(x_0y_0, x_0y_0)$ and $(y_0x_0, y_0x_0)$. It is not hard to see that this sloppyness is not a problem here.

**From domino to hhp-bisimulation.** Let $B \subseteq D_1 \times D_2 \times \mathbb{N} \times \mathbb{N}$ be a domino
bisimulation relating $T_1$ and $T_2$. We define a winning strategy for Duplicator in
game $B_{\text{hhp}}(\unf(A(T_1)), \unf(A(T_2)))$ in the following way.

If Spoiler makes a backward move then the response of Duplicator is determined
uniquely. Moreover, this response can always be performed because asynchronous
transition systems $\unf(A(T_1))$ and $\unf(A(T_2))$ have the property that every pair of
runs with equal labelling sequences has equal sets of most recent events. If Spoiler
makes a forward move by choosing an event $x_n$ or $y_m$, for $n, m \in \mathbb{N}$, then Duplicator
responds with the same event in the other transition system.

The only non-trivial responses of Duplicator are the ones to be made when Spoiler
makes a forward move by choosing an event of the form $d_{nm}$, where $d$ is a domino,
and $n, m \in \mathbb{N}$. We define these responses by referring to the domino bisimulation $B$.

The strategy for Duplicator we define below has the following property.

**Property 4.1.** Suppose that a configuration $(n, m, C_1, C_2)$ of an hhp-bisimilarity
checking game $B_{\text{hhp}}(\unf(A(T_1)), \unf(A(T_2)))$ can be reached from the initial configuration
while Duplicator is playing according to the strategy. Then $d_{nm} \in C_1$ and $\epsilon_{nm'} \in C_2$, for $n' \in \{n-1, n\}$ and $m' \in \{m-1, m\}$, imply that $(d, \epsilon, n', m') \in B$.

Suppose without loss of generality that Spoiler makes a move in $\unf(A(T_1))$;
the other case is symmetric. We consider several cases depending on the current
configuration of $B_{\text{hhp}}(\unf(A(T_1)), \unf(A(T_2)))$.

- The current configuration of the game $B_{\text{hhp}}(\unf(A(T_1)), \unf(A(T_2)))$ is

$$(n, m, \emptyset, \emptyset)$$

for some $n, m \in \mathbb{N}$. Spoiler can choose an event $d_{nm'}$, such that $n' \in \{n-1, n\}$ and
$m' \in \{m-1, m\}$. Then Duplicator responds with an event $\epsilon_{nm'}$ in $\unf(A(T_2))$, such that $(d, \epsilon, n', m') \in B$.

- The current configuration of the game $B_{\text{hhp}}(\unf(A(T_1)), \unf(A(T_2)))$ is

$$(n, m, \{d_{nm'}\}, \{\epsilon_{nm'}\})$$

such that $n' \in \{n-1, n\}$ and $m' \in \{m-1, m\}$. Spoiler can choose an event $d'_{k, \ell}$,
such that either $k = n'$ and $\{m', \ell\} = \{m-1, m\}$, or $\ell = m'$ and $\{n', k\} = \{n-1, n\}$. In both cases Duplicator responds with an event $\epsilon'_{k, \ell}$, such that configurations $(\epsilon, n', m')$ and $(d', \epsilon', k, \ell)$ of $T_2$ are compatible, and $(d', \epsilon', k, \ell) \in B$. 


Note that all the responses we have defined above are indeed possible due to Property 4.1 and the definition of a domino bisimulation, and moreover, they maintain Property 4.1.

**From hhp- to domino bisimulation.** Let $B$ be an hhp-bisimulation relating $\mathcal{U}_\text{hp}(A(T_1))$ and $\mathcal{U}_\text{hp}(A(T_2))$. We define a winning strategy for Duplicator in game $\mathcal{B}_d(T_1, T_2)$. The strategy for Duplicator we define below has the following property.

**Property 4.2.** If configuration $(d, e, n, m)$ of $\mathcal{B}_d(T_1, T_2)$ can be reached while Duplicator is playing according to the strategy then $(n, m, \{d_{nm}\}, \{e_{nm}\}) \in B$.

Suppose without loss of generality that Spoiler makes a move in $T_1$; the other case is symmetric. We consider the two kinds of moves possible in a domino bisimilarity game.

- In order to fix an initial configuration of $\mathcal{B}_d(T_1, T_2)$ Spoiler chooses a configuration $(d, e, n, m)$ of $T_1$. Note that for all $n, m \in \mathbb{N}$, we have that $(n, m, \emptyset, \emptyset) \in B$. Let $e_{nm}$ be Duplicator’s response if Spoiler makes a forward move in the game $\mathcal{B}_\text{hp}(A(T_1), A(T_2))$ by choosing event $d_{nm}$ in configuration $(n, m, \emptyset, \emptyset)$. Then we take $e$ to be Duplicator’s response to Spoiler’s choice of configuration $(d, e, n, m)$.

- Let $(d, e, n, m)$ be the current configuration of $\mathcal{B}_d(T_1, T_2)$. In a next move Spoiler can choose a configuration $(d', e', n', m')$ of $T_1$ compatible with $(d, e, n, m)$. We consider cases when $(n', m') = (n - 1, m)$ and $(n', m') = n, m + 1$; the other two cases are analogous. Note that by Property 4.2 we have that

$$ (n, m, \{d_{nm}\}, \{e_{nm}\}) \in B. \quad (1) $$

- Let $(n', m') = (n - 1, m)$. Since configurations $(d, n, m)$ and $(d', n - 1, m)$ of $T_1$ are compatible, by applying condition 2. of the definition of an hhp-bisimulation to (1) we get that there is a domino $e'$ of $T_2$, such that configurations $(e, n, m)$ and $(e', n - 1, m)$ of $T_2$ are compatible, and

$$ (n, m, \{d_{(n-1)m}\}, \{e'_{(n-1)m}\}) \in B. \quad (2) $$

- Let $(n', m') = (n, m + 1)$. By applying condition 2. of the definition of an hhp-bisimulation to (1) we get that

$$ (n, m + 1, \{d_{nm}\}, \{e_{nm}\}) \in B. \quad (3) $$

Since configurations $(d, n, m)$ and $(d', n, m + 1)$ of $T_1$ are compatible, by applying condition 2. of the definition of an hhp-bisimulation to (3) we get that there is a domino $e'$ of $T_2$, such that configurations $(e, n, m)$ and $(e', n, m + 1)$ of $T_2$ are compatible, and

$$ (n, m + 1, \{d_{nm}\}, \{e'_{nm}\}) \in B. \quad (4) $$
We define event $e'$ to be Duplicator’s response in $B_4(T_1, T_2)$ for Spoiler’s move consisting of choosing configuration $(d', n, m + 1)$ of $T_1$. By applying condition 4. of the definition of an hhp-bisimulation to (4) we get

$$(n, m + 1, \{d'_n(m+1)\}, \{e'_n(m+1)\}) \in B.$$ 

Note that all the responses we have defined above are indeed possible due to Property 4.2 and the definition of a domino bisimulation, and moreover, they maintain Property 4.2.

### 4.4. Unlabelled hhp-bisimilarity

In this subsection we prove Theorem 2.2, i.e., we show that undecidability of hhp-bisimilarity holds also for unlabelled asynchronous transition systems. For an unlabelled origin constrained tiling system $T$ we give an effective definition of an asynchronous transition system $A'(T)$, which is obtained by only a slight modification of the asynchronous transition system $A(T)$: we add new states $s_1, s_2, s_3, s_4, s_5$, and new transitions $(0, 1) \xrightarrow{c} s_1 \xrightarrow{c} s_2 \xrightarrow{c} s_3 \xrightarrow{c} s_4 \xrightarrow{c} s_5$, where $c$ is a new event. The independence relation of $A'(T)$ is the same as in $A(T)$. There is no labelling function in $A'(T)$.

**Lemma 4.2 (The reduction).**

If Spoiler has a winning strategy in the labelled hhp-bisimilarity checking game $B_{\text{hp}}(\text{Unf}(A(T_1)), \text{Unf}(A(T_2)))$ then he has also a winning strategy in the unlabelled game $B_{\text{hp}}(\text{Unf}(A'(T_1)), \text{Unf}(A'(T_2)))$.

The rest of this section is devoted to proving the above lemma from which Theorem 2.2 follows by Theorem 3.2 and the proof of Theorem 2.1.

For notational convenience, below we sometimes identify configurations of the games $B_{\text{hp}}(\text{Unf}(A(T_1)), \text{Unf}(A(T_2)))$ or $B_{\text{hp}}(\text{Unf}(A'(T_1)), \text{Unf}(A'(T_2)))$, respectively, with pairs of states of the transition systems $\text{Unf}(A(T_1))$ and $\text{Unf}(A(T_2))$, or from the transition systems $\text{Unf}(A'(T_1))$ and $\text{Unf}(A'(T_2))$, respectively. The following observation will be useful.

**Lemma 4.3.** Let $(n_1, m_1, C_1)$ and $(n_2, m_2, C_2)$ be states of $\text{Unf}(A'(T_1))$ and $\text{Unf}(A'(T_2))$, respectively, such that $|C_1| \neq |C_2|$. Then Spoiler has a winning strategy in $B_{\text{hp}}(\text{Unf}(A'(T_1)), \text{Unf}(A'(T_2)))$ from these states.

**Proof.** Note that $|C_1|, |C_2| \in \{0, 1, 2\}$. If $|C_i| = 0$ and $|C_{3-i}| \neq 0$, for $i \in \{1, 2\}$, then an infinite number of forward moves is possible from $(n_i, m_i, C_i)$, and only finitely many from the state $(n_{3-i}, m_{3-i}, C_{3-i})$. Hence Spoiler has a winning strategy in this case.

If $|C_1| > 0$, $|C_2| > 0$, and $|C_1| \neq |C_2|$ then a winning strategy for Spoiler is as follows. Assume without loss of generality that $|C_1| = 1$; then of course $|C_2| = 2$. Spoiler makes a backward move in the first component, such that the next configu-
ration consists of states \( (n_1, m_1, 0) \) and \( (n'_2, m'_2, C'_2) \), where \( |C'_2| \neq 0 \). By the preceding argument it follows that Spoiler has a winning strategy from this configuration.

The argument to prove Lemma 4.2 is as follows. Consider a winning strategy for Spoiler in the game \( B_{hbp}(\text{Unf}(A(T_1)), \text{Unf}(A(T_2))) \). A configuration of the unlabelled hlp-bisimilarity checking game \( B_{hbp}(\text{Unf}(A'(T_1)), \text{Unf}(A'(T_2))) \) is called admissible if the corresponding configuration in the labelled hlp-bisimilarity checking game \( B_{hbp}(\text{Unf}(A(T_1)), \text{Unf}(A(T_2))) \) belongs to the winning strategy for Spoiler, i.e., if this configuration is reachable by playing the strategy from the initial configuration \( (0, 0, 0, \emptyset) \). Obviously, the initial configuration is admissible. We define a strategy for Spoiler in the unlabelled game to be equal to the strategy in the labelled game in all admissible configurations. In a sequence of lemmas (Lemmas 4.4–4.7 below) we argue that in all admissible configurations, every response of Duplicator which leads to a non-admissible configuration enables Spoiler to win in a finite number of steps.

Lemma 4.4. Let \( (0, 0, \emptyset) \) and \( (0, 0, \emptyset) \) be states of \( \text{Unf}(A'(T_1)) \) and \( \text{Unf}(A'(T_2)) \), respectively. If Spoiler chooses the move \( (0, 0, \emptyset) \xrightarrow{s} (1, 0, 0) \) then Duplicator must answer with the same move in the other system; otherwise Spoiler can win in a finite number of steps. The similar holds if Spoiler chooses the move \( (0, 0, \emptyset) \xrightarrow{d} (0, 1, 0) \).

Proof. By Lemma 4.3 we know that Duplicator cannot choose any transition \( (0, 0, \emptyset) \xrightarrow{d} (0, 0, \{d_{k0}\}) \), for \( d \in D \). Suppose then that his choice was \( (0, 0, \emptyset) \xrightarrow{d} (0, 1, 0) \). We argue that Spoiler has a winning strategy now. He plays \( (0, 0, \emptyset) \xrightarrow{e} s_1 \). If Duplicator answers with some \( (n, m, \emptyset) \) then he loses because of similar arguments as in the proof of Lemma 4.3. In any other case (after performing this move) there are at most three possible forward moves for Duplicator, whereas Spoiler can perform another four forward moves. Therefore, Spoiler has a winning strategy.

Lemma 4.5. Note that \( (n, m, 0, \emptyset) \) is an admissible configuration in the unlabelled hlp-bisimilarity checking game \( B_{hbp}(\text{Unf}(A'(T_1)), \text{Unf}(A'(T_2))) \), for all \( n, m \in \mathbb{N} \). If Spoiler chooses the move \( (n, m, 0) \xrightarrow{s} (n + 1, m, 0) \) then Duplicator must answer with the same move in the other system; otherwise Spoiler can win in a finite number of steps. The similar holds if Spoiler chooses the move \( (n, m, 0) \xrightarrow{d} (n, m + 1, 0) \).

Proof. The special case when \( n = m = 0 \) is proved in Lemma 4.4. By Lemma 4.3 it follows that Duplicator cannot choose any transition \( (n, m, 0, \emptyset) \xrightarrow{d} (n, m, \{d_{n', m'}\}) \). Suppose that the response of Duplicator is \( (n, m, 0) \xrightarrow{d} (n, m + 1, 0) \). Assume without loss of generality that the last event in the run \( \tau \) leading to the state \( (n, m, \emptyset) \) was \( y_{m-1} \). Now, Spoiler can win immediately by taking a backwards move since \( n + m \in \text{MR}(\tau \cdot x_n) \) and \( n + m \notin \text{MR}(\tau \cdot y_m) \).

Lemma 4.6. Note that \( (n, m, 0, \emptyset) \) is an admissible configuration in the unlabelled hlp-bisimilarity checking game \( B_{hbp}(\text{Unf}(A'(T_1)), \text{Unf}(A'(T_2))) \), for all \( n, m \in \mathbb{N} \).
If Spoiler chooses a move
\[(n, m, \emptyset) \xrightarrow{d_{n',m'}} (n, m, \{d_{n',m'}\}) ,\]
for \(n' \in \{n-1, n\}\), \(m' \in \{m-1, m\}\), and \(d_{n',m'} \in E_D\), then Duplicator must answer with
\[(n, m, \emptyset) \xrightarrow{\epsilon_{n',m'}} (n, m, \{\epsilon_{n',m'}\}) \]
in the other system, for some \(\epsilon_{n',m'} \in E_{D_2}\); otherwise Spoiler can win in a finite number of steps.

**Proof.** By Lemma 4.3 Duplicator must respond with a move of the form
\[(n, m, \emptyset) \xrightarrow{\epsilon_{n',m'}} (n, m, \{\epsilon_{n',m'}\}) .\]
We show that Spoiler has a winning strategy in all cases where \(n' \neq n''\) or \(m' \neq m''\).

We use Lemma 4.3 without explicitly mentioning it.

- If \(n' = n - 1\) and \(m' = m - 1\) then Duplicator looses for any other choice of indices \(n''\) and \(m''\). The reason is that from the state \((n, m, \{d_{n',m'}\})\) no forward move that preserves the cardinality of the set \(\{d_{n',m'}\}\) is available, and instead from the state \((n, m, \{\epsilon_{n',m'}\})\), a forward move can be made by choosing either the event \(x_n\) or \(y_m\).

- If \(n' = n - 1\) and \(m' = m\) then two forward moves are possible for Spoiler from the state \((n, m, \{d_{n,m'}\})\) that preserve the cardinality of the set \(\{d_{n',m'}\}\). If Duplicator chooses a different index \(n''\) or \(m''\) then he can perform at most one forward move preserving the cardinality of the set \(\{\epsilon_{n',m'}\}\).

However, Spoiler can now perform a backwards move reaching the state \((n, m + 1, \emptyset)\) and Duplicator's only response is by the same backwards move, reaching the state \((n + 1, m, \emptyset)\). Spoiler has a winning strategy now, by the arguments from Lemma 4.5.

- The case where \(n' = n\) and \(m' = m - 1\) is similar to the previous one.

\[\blacksquare\]
Lemma 4.7. Let \((n,m,\{d_{n,m}\},\{e_{n,m}\})\) be an admissible configuration in the unlabelled hhp-bisimilarity checking game \(B_{hlp}(\text{Unf}(A'(T_1)), \text{Unf}(A'(T_2)))\). If Spoiler chooses a move
\[
(n,m,\{d_{n,m}\}) \xrightarrow{d_{n,m}} (n',m',\{d_{n',m'}\}),
\]
for some \(n' \in \{n-1,n\}, m' \in \{m-1,m\}\), and \(d_{n',m'} \in E_{D_1}\), then Duplicator must answer with
\[
(n,m,\{e_{n,m}\}) \xrightarrow{e_{n,m}} (n',m',\{e_{n',m'}\})
\]
in the other system, for some \(e_{n',m'} \in E_{D_2}\); otherwise Spoiler can win in a finite number of steps.

We omit the proof of this lemma; similar arguments can be used as in the proof of Lemma 4.6.

Observe that all the \(E_D\)-events in the labelled asynchronous transition system \(A(T)\) have the same label since \(T\) is by our assumption an unlabelled origin constrained tiling system. Therefore, Lemmas 4.4-4.7 cover all the relevant cases in the argument for Lemma 4.2 sketched before Lemma 4.4. This concludes the reduction of unlabelled domino bisimilarity to unlabelled hhp-bisimilarity, and hence Theorem 2.2 follows from Theorem 3.2.

4.5. Finite elementary net system \(N(T)\)

In this subsection we argue that undecidability of hhp-bisimilarity for finite elementary net systems follows as a corollary of our proof for finite asynchronous transition systems.

Given a tiling system \(T\) we define an elementary net system \(N(T)\) and we argue that \(A(T)\) is isomorphic to the asynchronous transition system \(na(N(T))\) corresponding to the net \(N(T)\); see the articles by Nielsen and Winskel [30, 23] for the definition of the asynchronous transition system \(na(N(T))\). This immediately implies the following facts.

Theorem 4.1. Hhp-bisimilarity is undecidable for finite labelled elementary net systems.

Theorem 4.2. Hhp-bisimilarity is undecidable for finite unlabelled elementary net systems.

The elementary net system \(N(T) = (P_{N(T)},E_{N(T)},\text{pre}_{N(T)},\text{post}_{N(T)},M_{N(T)})\) is shown in Figure 2 and it consists of the following:

- the set of conditions

\[
P_{N(T)} = \{ a_i, b_i : i \in \{0,1,2,3,4\} \} \cup \{ a_{i+1}^j, b_{j+1}^i : i,j \in \{0,1,2,3\} \} \cup E_D;
\]
Variables $i$ and $j$ range over $\{0, 1, 2, 3\}$. Therefore we have four copies of each place in the second column, for $j \in \{0, 1, 2, 3\}$, and four copies of each place in the third column, for $i \in \{0, 1, 2, 3\}$.

The preconditions of each $d_{ij} \in E_D$, are the two places $a^j_{i(i+1)}$ and $b^i_{j(j+1)}$, and all the places in $\text{Inempt}(d_{ij}) = \{ \epsilon_{k\ell} \in E_D : (d_{ij}, \epsilon_{k\ell}) \notin I_D \}$.

**Figure 2.** The elementary net system $N(T)$. 
• the set of events \( E_{N(T)} = E_{A(T)} \);
• the function \( \text{pre}_{N(T)} : E_{N(T)} \to \wp(P_{N(T)}) \) specifying the set of places in the pre-condition of an event:

\[
\text{pre}_{N(T)}(e) = \begin{cases} 
\{ a_0 \} & \text{if } e = x_0, \\
\{ b_0 \} & \text{if } e = y_0, \\
\{ a_i \} \cup A_{(i-1)i} & \text{if } e = x_i \text{ for } i \in \{ 1, 2, 3, 4 \}, \\
\{ b_j \} \cup B_{(j-1)j} & \text{if } e = y_j \text{ for } j \in \{ 1, 2, 3, 4 \}, \\
\{ a_{i+1}^j \}, b_{(j+1)i} & \text{if } e = \text{Incomp}(d_{ij}) \text{ for } d_{ij} \in E_D,
\end{cases}
\]

where for \( i, j \in \{ 0, 1, 2, 3 \} \), we define \( A_{(i+1)i} = \{ a_k^{(i+1)} \} \) and \( B_{(j+1)j} = \{ b_k^{(j+1)} \} \), and for \( d_{ij} \in E_D \), we define the set of dominoes incompatible with \( d_{ij} \) by:

\[
\text{Incomp}(d_{ij}) = \{ \ e_{kl} \in E_D : (d_{ij}, e_{kl}) \notin I_D \} ;
\]

• the function \( \text{post}_{N(T)} : E_{N(T)} \to \wp(P_{N(T)}) \) specifying the set of places in the post-condition of an event:

\[
\text{post}_{N(T)}(e) = \begin{cases} 
\{ a_{i+1} \} \cup A_{(i+1)(i+2)} & \text{if } e = x_i \text{ for } i \in \{ 0, 1, 2 \}, \\
\{ a_4 \} \cup A_{12} & \text{if } e = x_3, \\
\{ a_2 \} \cup A_{23} & \text{if } e = x_4, \\
\{ b_{j+1} \} \cup B_{(j+1)(j+2)} & \text{if } e = y_j \text{ for } j \in \{ 0, 1, 2 \}, \\
\{ b_4 \} \cup B_{12} & \text{if } e = y_3, \\
\{ b_2 \} \cup B_{23} & \text{if } e = y_4, \\
0 & \text{if } e \in E_D;
\end{cases}
\]

The asynchronous transition system \( A(T) \) is isomorphic to \( \nu(P_{N(T)}) \).

**Proposition 4.4.** The asynchronous transition system \( A(T) \) is isomorphic to \( \nu(P_{N(T)}) \).

**Proof.** We define a function \( \Xi : S_{A(T)} \to \wp(P_{N(T)}) \) as follows:

\[
\Xi((i, j, C)) = \{ a_i, b_j \} \cup X_i \cup Y_j \cup E_D \setminus \bigcup_{e \in C} \text{pre}_{N(T)}(e),
\]

where

\[
X_i = \begin{cases} 
A_{01} & \text{if } i = 0, \\
A_{(i-1)i} \cup A_{(i+1)} & \text{if } i \in \{ 1, 2, 3 \}, \\
A_{34} \cup A_{12} & \text{if } i = 4,
\end{cases}
\]

and similarly

\[
Y_j = \begin{cases} 
B_{01} & \text{if } j = 0, \\
B_{(j-1)j} \cup B_{(j+1)} & \text{if } j \in \{ 1, 2, 3 \}, \\
B_{34} \cup B_{12} & \text{if } j = 4.
\end{cases}
\]
In order to argue that $\Xi$ is an isomorphism of asynchronous transition systems $A(T)$ and $na(N(T))$ it suffices to establish the following:

1. $\Xi(s^\text{ini}_{A(T)}) = M_{N(T)}$, i.e., the initial state of $A(T)$ is mapped by $\Xi$ to the initial marking of $N(T)$, 
2. for all $s \in S_{A(T)}$, 
   (i) if $s \xrightarrow{\zeta} A(T) t$ then $\Xi(s) \xrightarrow{\zeta} na(N(T)) \Xi(t)$, 
   (ii) if $\Xi(s) \xrightarrow{\zeta} na(N(T)) M$ then there is $t \in S_{A(T)}$, such that $s \xrightarrow{\zeta} A(T) t$ and $M = \Xi(t)$, 
3. $(e, f) \in I_{A(T)}$ if and only if $\bullet e \bullet \cap \bullet f \bullet = \emptyset$.

It is a tedious but routine exercise to verify that clauses 1–3. hold. ■

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