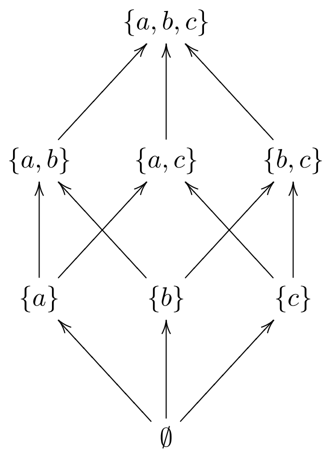
Tutorial 6 - Solutions

Exercise 1*

Draw a graphical representation of the complete lattice $(2^{\{a,b,c\}}, \subseteq)$ and compute supremum and infimum of the sets below.

The complete lattice:



- $\sqcap\{\{a\},\{b\}\} = \emptyset$
- $\sqcup\{\{a\},\{b\}\} = \{a,b\}$
- $\sqcap\{\{a\},\{a,b\},\{a,c\}\} = \{a\}$
- $\sqcup\{\{a\},\{a,b\},\{a,c\}\} = \{a,b,c\}$
- $\sqcap\{\{a\},\{b\},\{c\}\} = \emptyset$
- $\sqcup\{\{a\},\{b\},\{c\}\} = \{a,b,c\}$
- $\sqcap\{\{a\},\{a,b\},\{b\},\emptyset\} = \emptyset$
- $\sqcup\{\{a\},\{a,b\},\{b\},\emptyset\} = \{a,b\}$

Exercise 2

Prove that for any partially ordered set $(D, \sqsubseteq)$ and any $X \subseteq D$, if supremum of X ($\sqcup X$) and infimum of X ($\sqcap X$) exist then they are uniquely defined. (Hint: use the definition of supremum and infimum and antisymmetry of $\sqsubseteq$.)

We prove the claim for the supremum (least upper bound) of X. The arguments for the infimum are symmetric. Let $d_1, d_2 \in D$ be two supremums of a given set X. This means that $X \sqsubseteq d_1$ and $X \sqsubseteq d_2$ as both d_1 and d_2 are upper bounds of X. Now because d_1 is the least upper bound and d_2 is an upper bound we get $d_1 \sqsubseteq d_2$. Similarly, d_2 is the least upper bound and d_1 is an upper bound so $d_2 \sqsubseteq d_1$. However, from antisymmetry and $d_1 \sqsubseteq d_2$ and $d_2 \sqsubseteq d_1$ we get that $d_1 = d_2$.

Exercise 3

Let $(D, \sqsubseteq)$ be a complete lattice. What are $\sqcup\emptyset$ and $\sqcap\emptyset$ equal to?

- $\sqcup\emptyset = \bot = \sqcap D$.
- $\sqcap\emptyset = \top = \sqcup D$.

Exercise 4*

Consider the complete lattice $(2^{\{a,b,c\}}, \subseteq)$. Define a function $f : 2^{\{a,b,c\}} \rightarrow 2^{\{a,b,c\}}$ such that f is monotonic.

For example we define f as follows (note that there are many possibilites):

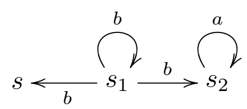
S	$f(S)$
$\emptyset$	$\{a\}$
$\{a\}$	$\{a\}$
$\{b\}$	$\{a\}$
$\{c\}$	$\{a\}$
$\{a,b,c\}$	$\{a,b\}$
$\{a,b\}$	$\{a,b\}$
$\{a,c\}$	$\{a,b\}$
$\{b,c\}$	$\{a,b\}$

The function f is monotonic which we can verify by a case inspection.

- Compute the greatest fixed point by using directly the Tarski's fixed point theorem.
  - According to Tarski's fixed point theorem the largest fixed point z_{max} is given by $z_{\text{max}} = \sqcup A$, where
  $$A = \{x \in 2^{\{a,b,c\}} \mid x \sqsubseteq f(x)\}.$$
  In our case, by the definition of f we get $A = \{\emptyset, \{a\}, \{a,b\}\}$. The supremum of A in $2^{\{a,b,c\}}$ is $\{a,b\}$ so by Tarski's fixed point theorem, the largest fixed point of f is $\{a,b\}$.
- Compute the least fixed point by using the Tarski's fixed point theorem for finite lattices (i.e. by starting from $\perp$ and by applying repeatedly the function f until the fixed point is reached).
  - First note that $\perp = \sqcap 2^{\{a,b,c\}} = \emptyset$. We now repeatedly apply f until it stabilizes
  $$f(\emptyset) = \{a\}$$
  $$f(f(\emptyset)) = f(\{a\}) = \{a\}$$
  and hence the least fixed point of f is $\{a\}$.

Exercise 5

Consider the following labelled transition system.



Compute for which sets of states $[\![X]\!] \subseteq \{s, s_1, s_2\}$ the following formulae are true.

- $X = \langle a \rangle tt \vee [b]X$
  - The equation holds for the following sets of states: $\{s_2, s\}, \{s_2, s_1, s\}$.
- $X = \langle a \rangle tt \vee ([b]X \wedge \langle b \rangle tt)$
  - The equation holds only for the set $\{s_2\}$.

Exercise 6 (optional)

Exercise A.6, part 2. on page 91 in *A note of Milner's CCS*.