Comparing the speed of probabilistic processes

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Semi-Markov processes (SMPs) are real-time stochastic processes in which the time that is spent in a given state before a transition is fired is determined by an arbitrary probability distribution.

Continuous-time Markov chains are a special case of SMPs where all residence-time distributions are exponential.

SMPs have been used extensively to model systems where the time that is spent in a given state before a transition is fired is not exponentially distributed.

The faster-than relation can not be approximated up to a multiplicative constant.

Definition 1. s is faster than s' if for any word w and time point t, s has a higher probability of outputting w within time t than s' does.

Example 1. Consider the SMP in Fig. 2. Let

be random variables. Denote by \( P_s(w, X \leq t) \) the probability that s outputs the word w and the random variable X is less than or equal to t.

\[
P_s(a, X_1 \leq t) \geq P_{s'}(a, X_2 \leq t)
\]

because X1 has a higher rate than X2.

\[
P_s(aa, X_1 + X_2 \leq t) = P_s(aa, X_2 + X_1 \leq t) = P_{s'}(aa, X_2 + X_1 \leq t)
\]

because addition of random variables is commutative, and hence also

\[
P_s(aaa^\omega, X_1 + X_2 + X_3^\omega \leq t) = P_{s'}(aaa, X_2 + X_1 + X_3^\omega \leq t).
\]

Therefore s is faster than s'.

Hardness results

Through a connection to the Universality problem for probabilistic automata, we obtain the following undecidability result.

Theorem 1. The faster-than relation is undecidable.

Since the Universality problem with only one label is equivalent to the Positivity problem for linear recurrence sequences (which is related to the Skolem problem) [1], we also get a hardness result.

Theorem 2. The faster-than relation is Positivity-hard for only one label.

Furthermore, utilising a celebrated theorem for probabilistic automata by Condon and Lipton [2], we have an inapproximability result.

Theorem 3. The faster-than relation can not be approximated up to a multiplicative constant.

Unambiguous processes

A SMP is unambiguous if every output label leads to a unique successor state.

Example 2. The SMP in Fig. 1 is not unambiguous, since \( s_u \) has a-transitions to both \( s_u \) and \( s_v \), whereas the SMP in Fig. 2 is unambiguous.

For unambiguous SMPs we can recover decidability.

Theorem 4. For unambiguous SMPs, the faster-than relation is decidable in coNP.

Given a state space S and states s and s', the algorithm is as follows:

- Using a simple graph analysis, find all the states p and p' reachable from s and s' such that there is a looping word w that takes p to p and p' to p'.
- Check whether s is faster than s' for all words of length less than \(|S|^2\) and p is faster than p' for all looping words of length less than \(|S|^2\).

Example 3. Consider the SMP in Fig. 3.

We want to check whether \( s_1 \) is faster than \( s_2 \).

- From \((s_1, s_2)\) we can reach \((s_3, s_4)\) which has looping words of the form \( b^ab^ab\) a, and we can also reach \((s_5, s_6)\) with the same looping words.
- One can easily check that \( s_1 \) is faster than \( s_2 \) and that \( s_5 \) is faster than \( s_6 \) and vice versa for all looping words.
- However, we can also see that \( s_1 \) is faster than \( s_2 \) if and only if \( s_5 \) and \( s_6 \) have the same rate, for if \( s_5 \) had a higher rate, then the looping word \( aba \) would have a higher probability in \( s_2 \) than in \( s_1 \). Likewise, if \( s_6 \) had a higher rate, the looping word \( baa \) would fail the check.

Time-bounded approximation

Under the following assumptions, we can recover approximability:

- approximation up to an additive constant,
- consider only time up to a given time bound,
- slow distributions that must use some amount of time.

Under these assumptions we get the following:

- Slow distributions must use some non-zero amount of time in each step, so long words have a very small probability of being output within the given time bound.
- Hence words above a given length will have probability less than the desired approximation accuracy.
- We therefore need only check words up to a given length.

Theorem 5. Approximating the time-bounded faster-than relation up to an additive constant is possible.

Open problems

There are a number of open problems still:

- The symmetric equally-fast relation.
- Reactive models instead of generative models.
- Logical aspects of the relation.
- Compositional aspects of the relation, including timing anomalies.

References