Syntax and semantics - Mathematical Background
(Homework 1)

1 Readings

Sipser's book: chapter 0 – Introduction – reflect on section 0.4 – Types of Proof

Hüttel's book: Part I – Background

2 Exercises

Exercise 1. Describe the following sets:

(a) $\{1, 2, 3\} \times \{a, b\}$

(b) $\mathcal{P}(\{1, 2, 3\}) \times \{a\}$

(c) $\mathcal{P}(\emptyset) \setminus \emptyset$

(d) $\mathcal{P}(\{\emptyset, \{\emptyset\}\})$

(e) $\mathcal{P}(\{a, b, \{a\}, \{a, b\}\})$

(f) $\overline{\{a, b, c\} \setminus \{c, d\}}$, where $U = \{a, b, c, d\}$.

Exercise 2. Prove the following equalities for arbitrary sets $A, B \subseteq U$:

(a) $\overline{A \cup B} = \overline{A} \cap \overline{B}$

(b) $\overline{A \cap B} = \overline{A} \cup \overline{B}$

Exercise 3. Prove the following statements:

(a) the function $f\colon \mathbb{Z} \to \mathbb{Z}$, $f(x) = x + 2$ is bijective;

(b) the function $f\colon \mathbb{N} \to \mathbb{N}$, $f(x) = x + 2$ is not bijective;

(c) the function $f\colon \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} x+1 & \text{if } x \leq 5 \\ x+2 & \text{otherwise} \end{cases}$$

is injective but not surjective.

Exercise 4. Consider the binary relation $\sqsubseteq\ \subseteq (\mathbb{R} \times \mathbb{R}) \times (\mathbb{R} \times \mathbb{R})$ defined as follows

$$(x_1, x_2) \sqsubseteq (y_1, y_2) \quad \text{if} \quad x_1 \leq y_1 \text{ and } x_2 \leq y_2 \,.$$

(a) Is $\sqsubseteq$ symmetric?

(b) Is $\sqsubseteq$ antisymmetric?

(c) Is $\sqsubseteq$ reflexive?

(d) Is $\sqsubseteq$ transitive?

(e) Is $\sqsubseteq$ an equivalence relation?

Exercise 5. Prove, using the truth tables, that the following boolean expressions are equal

(a) $a \to b$ and $(\neg a) \vee b$

(b) $(a \to (b \to c))$ and $(a \wedge b) \to c$

(c) $a \leftrightarrow b$ and $(a \wedge b) \vee ((\neg a) \wedge (\neg b))$

(d) $\neg(a \wedge b)$ and $(\neg a) \vee (\neg b)$

(e) $\neg(a \vee b)$ and $(\neg b) \wedge (\neg b)$

Exercise 6. Given the alphabet $\Sigma = \{a, b\}$, we define the prefix order $u \sqsubseteq v$ for arbitrary $u, v \in \Sigma^*$ by

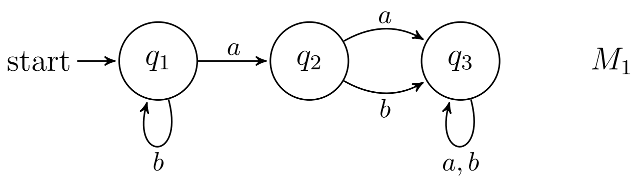
$$u \sqsubseteq v \quad \text{if there exists } w \in \Sigma^* \text{ such that } uw = v.$$

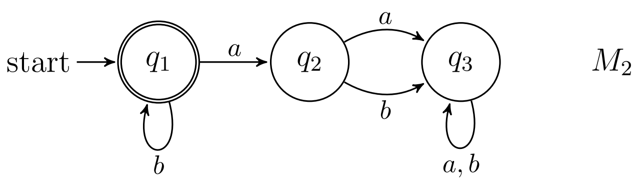
(a) Find two strings u and v such that $u \sqsubseteq v$.

(b) Prove that $\sqsubseteq$ is reflexive, antisymmetric and transitive.

(c) Give an example to show that $\sqsubseteq$ is not symmetric.

Exercise 7. Give a formal description of the following machines:





(a) Describe the sequence of states of M_1 for the following inputs

$$abbbab\,, \quad ababaab\,, \quad aaaaa\,, \quad \epsilon\,.$$

(b) Describe the words accepted by each of the two machines.

(c) Modify the machine M_1 such that it will accept any word in the language $\{a, b\}$.

Exercise 8. The formal description of a DFA M is

$$(\{q_1, q_2, q_3, q_4, q_5\}, \{u, v\}, \delta, q_3, \{q_3\})\,,$$

where δ is given by the table below

	u	v
q_1	q_1	q_2
q_2	q_1	q_3
q_3	q_2	q_4
q_4	q_3	q_5
q_5	q_4	q_5

Give the state diagram of this machine.