

# Verification, Performance Analysis and Controller Synthesis for Real-Time Systems

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## Abstract.

This note aims at providing a concise and precise *Travellers Guide*, *Phrase Book* or *Reference Manual* to the timed automata modeling formalism introduced by Alur and Dill [7,8]. The note gives comprehensive definitions of timed automata, priced (or weighted) timed automata, and timed games and highlights a number of results on associated decision problems related to model checking, equivalence checking, optimal scheduling, and winning strategies.

**Keywords.** Timed automata, regions, zones, reachability, bisimilarity; priced and weighted timed automata, optimal reachability, optimal infinite runs, conditional optimality; timed games, winning strategies.

## 1. Introduction

The model of timed automata, introduced by Alur and Dill [7,8], has by now established itself as a classical formalism for describing the behaviour of real-time systems. A number of important properties has been shown decidable, including reachability, model checking and several behavioural equivalences and preorders.

By now, real-time model checking tools such as UPPAAL [17,57] and KRONOS [32] are based on the timed automata formalism and on the substantial body of research on this model that has been targeted towards transforming the early results into practically efficient algorithms — *e.g.* [14,21,13,19] — and data structures — *e.g.* [56,54,20,20].

The maturity of a tool like UPPAAL is witnessed by the numerous applications — *e.g.* [48,60,55,45,40,38,52,61] — to the verification of industrial case-studies spanning real-time controllers and real-time communication protocols. More recently, model-checking tools in general and UPPAAL in particular have been applied to solve realistic scheduling problems by a reformulation as reachability problems — *e.g.* [42,47,1,62].

Aiming at performance analysis allowing to formulate and solve *optimal* scheduling problems, a recent extension of timed automata is that of *priced* or

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*weighted* timed automata [18,9]. Surprisingly, a number of properties have been shown to be decidable for this formalism [41,58,18,9,29]. The newly emerged branch UPPAAL CORA provides an efficient tool for solving cost-optimal reachability problems [53] and has been applied successfully to a number of optimal scheduling problems, *e.g.* [15,22,44].

Most recently, substantial efforts have been made on the automatic synthesis of (correct-by-construction) controllers from timed games for given control objectives. From early decidability results [63,12] the effort has led to efficient on-the-fly algorithms [68,34] with the newest branch UPPAAL TIGA [16] providing an efficient tool implementation with industrial applications emerging, *e.g.* [50].

This note aims at providing a concise and precise *Travellers Guide, Phrase Book* or *Reference Manual* to the land and language of timed automata. The note gives comprehensive definitions of timed automata, weighted timed automata, and timed games and highlights a number of results on associated decision problems related to model checking, equivalence checking, optimal scheduling, and winning strategies. The intention is that the note should provide an easy-to-access introduction and overview of the field to anyone interested.

## 2. Timed automata

In this section we review the notion of timed automata introduced by Alur and Dill [7,8] as a formalism for describing the behaviour of real-time systems. We review the syntax and semantics and highlight the by now classical region construction underlying the decidability of several associated problems.

Here we illustrate how regions are applied in showing decidability of reachability and timed and untimed (bi)similarity. However, though indispensable for showing decidability, the notion of region does not provide the means for efficient, practical tool implementations. The verification engine of UPPAAL instead applies so-called zones, which are *convex unions* of regions. We give a brief account of zones as well as their efficient representation and manipulation using difference-bound matrices.

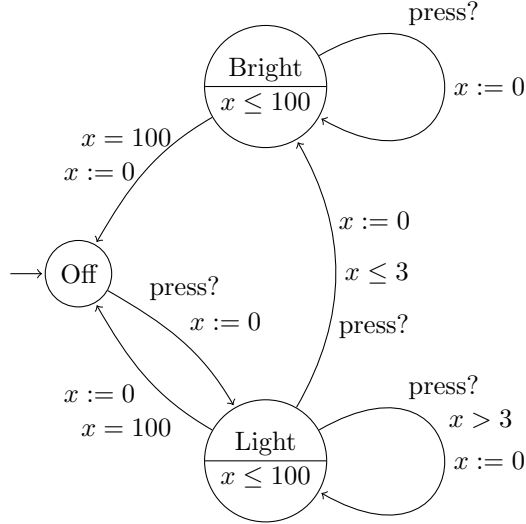
### 2.1. Syntax and semantics

**Definition 2.1.** The set  $\Phi(C)$  of *clock constraints*  $\varphi$  over a finite set (of *clocks*)  $C$  is defined by the grammar

$$\varphi ::= x \bowtie k \mid \varphi_1 \wedge \varphi_2 \quad (x \in C, k \in \mathbb{Z}, \bowtie \in \{\leq, <, \geq, >\})$$

The set  $\Phi^+(C)$  of *extended clock constraints*  $\varphi$  is defined by the grammar

$$\varphi ::= x \bowtie k \mid x - y \bowtie k \mid \varphi_1 \wedge \varphi_2 \quad (x, y \in C, k \in \mathbb{Z}, \bowtie \in \{\leq, <, \geq, >\})$$



**Figure 1.** A light switch modelled as a timed automaton.

**Remark 2.2.** The clock constraints in  $\Phi(C)$  above are also called *diagonal-free* clock constraints, and the additional ones in  $\Phi^+(C)$  are called *diagonal*. We restrict ourselves to diagonal-free clock constraints here; see Remark 2.44 for one reason. For additional modelling power, timed automata with diagonal constraints can be used, as it is shown in [8,26] that any such can be converted to a diagonal-free one; however the conversion may lead to an exponential blow-up.

**Definition 2.3.** A *timed automaton* is a tuple  $(L, \ell_0, F, C, \Sigma, I, E)$  consisting of a finite set  $L$  of locations, an initial location  $\ell_0 \in Q$ , a set  $F \subseteq Q$  of final locations, a finite set  $C$  of clocks, a finite set  $\Sigma$  of actions, a location invariants mapping  $I : L \rightarrow \Phi(C)$ , and a set  $E \subseteq L \times \Phi(C) \times \Sigma \times 2^C \times L$  of edges.

We shall denote an edge  $(\ell, \varphi, a, r, \ell') \in E$  by  $\ell \xrightarrow{\varphi, a, r} \ell'$ .

**Example 2.1.** Figure 1 provides a timed automaton model of an “intelligent” light switch. A press of the button turns on the light, which switches itself off after 100 time units. The switching-off can be prevented by pressing the button again, and pressing the button twice, with at most three time units between the presses, triggers a special bright light.

**Definition 2.4.** A *clock valuation* on a finite set  $C$  of clocks is a mapping  $v : C \rightarrow \mathbb{R}_{\geq 0}$ . The *initial* valuation  $v_0$  is given by  $v_0(x) = 0$  for all  $x \in C$ . For a valuation  $v$ ,  $d \in \mathbb{R}_{\geq 0}$ , and  $r \subseteq C$ , the valuations  $v + d$  and  $v[r]$  are defined by

$$(v + d)(x) = v(x) + d$$

$$v[r](x) = \begin{cases} 0 & \text{for } x \in r \\ v(x) & \text{for } x \notin r \end{cases}$$

**Note.** We will in general write  $B^A$  for the set of mappings from a set  $A$  to a set  $B$ . The set of clock valuations on  $C$  is thus  $\mathbb{R}_{\geq 0}^C$ .

**Definition 2.5.** The *zone* of an extended clock constraint in  $\Phi(C)$  is a set of clock valuations  $C \rightarrow \mathbb{R}_{\geq 0}$  given inductively by

$$\begin{aligned} \llbracket x \bowtie k \rrbracket &= \{v : C \rightarrow \mathbb{R}_{\geq 0} \mid v(x) \bowtie k\} \\ \llbracket x - y \bowtie k \rrbracket &= \{v : C \rightarrow \mathbb{R}_{\geq 0} \mid v(x) - v(y) \bowtie k\} \\ \llbracket \varphi_1 \wedge \varphi_2 \rrbracket &= \llbracket \varphi_1 \rrbracket \cap \llbracket \varphi_2 \rrbracket \end{aligned}$$

We shall write  $v \models \varphi$  instead of  $v \in \llbracket \varphi \rrbracket$ .

**Definition 2.6.** The *semantics* of a timed automaton  $A = (L, \ell_0, F, C, \Sigma, I, E)$  is the transition system  $\llbracket A \rrbracket = (S, s_0, \Sigma \cup \mathbb{R}_{\geq 0}, T = T_s \cup T_d)$  given by

$$\begin{aligned} S &= \{(\ell, v) \in L \times \mathbb{R}_{\geq 0}^C \mid v \models I(\ell)\} & s_0 &= (\ell_0, v_0) \\ T_s &= \{(\ell, v) \xrightarrow{a} (\ell', v') \mid \exists \ell' \xrightarrow{\varphi, a, r} \ell' \in E : v \models \varphi, v' = v[r]\} \\ T_d &= \{(\ell, v) \xrightarrow{d} (\ell, v + d) \mid \forall d' \in [0, d] : v + d' \models I(\ell)\} \end{aligned}$$

**Remark 2.7.** The transition system  $\llbracket A \rrbracket$  from above is an example of what is known as a *timed transition system*, *i.e.* a transition system where the label set includes  $\mathbb{R}_{\geq 0}$  as a subset and which satisfies certain additivity and time determinacy properties. We refer to [2] for a more in-depth treatment.

**Definition 2.8.** A (finite) *run* of a timed automaton  $A = (L, \ell_0, F, C, \Sigma, I, E)$  is a finite path  $\rho = (\ell_0, v_0) \rightarrow \dots \rightarrow (\ell_k, v_k)$  in  $\llbracket A \rrbracket$ .  $\rho$  is said to be *accepting* if  $\ell_k \in F$ .

**Example 2.1 (continued).** The light switch model from figure 1 has as state set

$$S = \{\text{Off}\} \times \mathbb{R}_{\geq 0} \cup \{\text{Light}, \text{Bright}\} \times [0, 100]$$

where we identify valuations with their values at  $x$ . A few example runs are given below; we abbreviate “press?” to “p”:

$$\begin{aligned} (\text{Off}, 0) &\xrightarrow{150} (\text{Off}, 150) \xrightarrow{\text{p}} (\text{Light}, 0) \xrightarrow{100} (\text{Light}, 100) \rightarrow (\text{Off}, 0) \\ (\text{Off}, 0) &\xrightarrow{\text{p}} (\text{Light}, 0) \xrightarrow{10} (\text{Light}, 10) \xrightarrow{\text{p}} (\text{Light}, 0) \xrightarrow{100} (\text{Light}, 100) \rightarrow (\text{Off}, 0) \\ (\text{Off}, 0) &\xrightarrow{\text{p}} (\text{Light}, 0) \xrightarrow{1} (\text{Light}, 1) \xrightarrow{\text{p}} (\text{Bright}, 0) \xrightarrow{100} (\text{Bright}, 100) \rightarrow (\text{Off}, 0) \end{aligned}$$

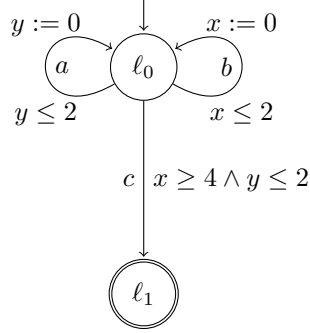


Figure 2. A timed automaton with two clocks.

**Example 2.2** (cf. [2, Ex. 11.7]). Figure 2 shows a timed automaton  $A$  with two clocks and a final location  $\ell_1$ . In the next section we shall be concerned with a *reachability* problem: Is the location  $\ell_1$  reachable, *i.e.* does there exist a state  $(\ell_1, v)$  in  $\llbracket A \rrbracket$  for some valuation  $v$ ?

## 2.2. Reachability

We are concerned with the following problem: Given a timed automaton  $A = (L, \ell_0, F, C, \Sigma, I, E)$ , is any of the locations in  $F$  reachable? We shall later define the *timed language* generated by a timed automaton and see that this reachability problem is equivalent to *emptiness checking*: Is the timed language generated by  $A$  non-empty?

An immediate obstacle to reachability checking is the infinity of the state space of  $A$ : The transition system  $\llbracket A \rrbracket$  has uncountably many states, hence straight-forward naïve reachability algorithms do not work for us.

**Notation 2.9.** The *derived transition relations* in a timed automaton  $A = (L, \ell_0, F, C, \Sigma, I, E)$  are defined as follows: For  $(\ell, v), (\ell', v')$  states in  $\llbracket A \rrbracket$ , we say that

- $(\ell, v) \xrightarrow{\delta} (\ell', v')$  if  $(\ell, v) \xrightarrow{d} (\ell', v')$  in  $\llbracket A \rrbracket$  for some  $d > 0$ ,
- $(\ell, v) \xrightarrow{\alpha} (\ell', v')$  if  $(\ell, v) \xrightarrow{a} (\ell', v')$  in  $\llbracket A \rrbracket$  for some  $a \in \Sigma$ , and
- $(\ell, v) \rightsquigarrow (\ell', v')$  if  $(\ell, v) \xrightarrow{(\delta \cup \alpha)^*} (\ell', v')$ .

**Definition 2.10.** The set of *reachable locations* in a timed automaton  $A = (L, \ell_0, F, C, \Sigma, I, E)$  is

$$\text{Reach}(A) = \{\ell \in L \mid \exists v : C \rightarrow \mathbb{R}_{\geq 0} : (\ell_0, v_0) \rightsquigarrow (\ell, v)\}$$

Hence we can now state the reachability problem as follows:

**Problem 2.1 (Reachability).** Given a timed automaton  $A = (L, \ell_0, F, C, \Sigma, I, E)$ , is  $\text{Reach}(A) \cap F \neq \emptyset$ ?

**Definition 2.11.** Let  $A = (L, \ell_0, F, C, \Sigma, I, E)$  be a timed automaton. A reflexive relation  $R \subseteq L \times \mathbb{R}_{\geq 0}^C \times L \times \mathbb{R}_{\geq 0}^C$  is a *time-abstracted simulation* provided that for all  $(\ell_1, v_1) R (\ell_2, v_2)$ ,

- for all  $(\ell_1, v_1) \xrightarrow{\delta} (\ell'_1, v'_1)$  there exists  $(\ell'_2, v'_2) R (\ell_1, v_1)$  for which  $(\ell_2, v_2) \xrightarrow{\delta} (\ell'_2, v'_2)$ , and
- for all  $(\ell_1, v_1) \xrightarrow{a} (\ell'_1, v'_1)$ ,  $a \in \Sigma$ , there exists  $(\ell'_2, v'_2) R (\ell_1, v_1)$  for which  $(\ell_2, v_2) \xrightarrow{a} (\ell'_2, v'_2)$ .

$R$  is called a *time-abstracted bisimulation* if it is also symmetric; it is said to be *F-sensitive* if additionally,  $(\ell_1, v_1) R (\ell_2, v_2)$  implies that  $\ell_1 \in F$  if and only if  $\ell_2 \in F$ .

Note that a time-abstracted (bi)simulation on  $A$  is the same as a standard (bi)simulation on the transition system derived from  $\llbracket A \rrbracket$  with transitions  $\xrightarrow{\delta}$  and  $\xrightarrow{a}$ . Likewise, the quotient introduced below is just the bisimulation quotient of that derived transition.

**Definition 2.12.** Let  $A = (L, \ell_0, F, C, \Sigma, I, E)$  be a timed automaton and  $R \subseteq L \times \mathbb{R}_{\geq 0}^C \times L \times \mathbb{R}_{\geq 0}^C$  a time-abstracted bisimulation. The *quotient* of  $\llbracket A \rrbracket = (S, s_0, \Sigma \cup \mathbb{R}_{\geq 0}, T)$  with respect to  $R$  is the transition system  $\llbracket A \rrbracket_R = (S_R, s_R^0, \Sigma \cup \{\delta\}, T_R)$  given by  $S_R = S/R$ ,  $s_R^0 = \langle s_0 \rangle_R$ , and with transitions

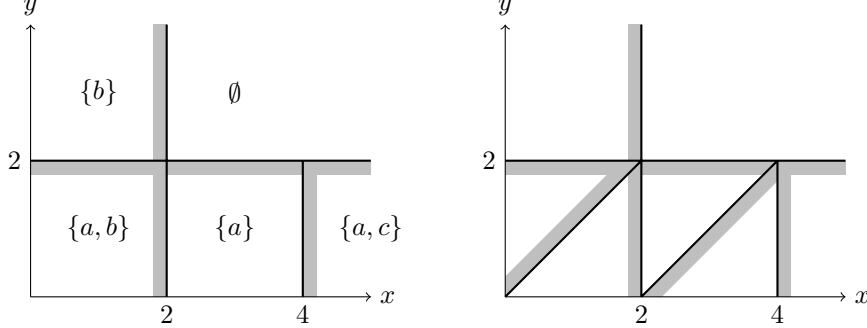
- $\pi \xrightarrow{\delta} \pi'$  whenever  $(\ell, v) \xrightarrow{\delta} (\ell', v')$  for some  $(\ell, v) \in \pi$ ,  $(\ell', v') \in \pi'$ , and
- $\pi \xrightarrow{a} \pi'$  whenever  $(\ell, v) \xrightarrow{a} (\ell', v')$  for some  $(\ell, v) \in \pi$ ,  $(\ell', v') \in \pi'$ .

The following proposition expresses that  $F$ -sensitive quotients are sound and complete with respect to reachability.

**Proposition 2.13 ([4]).** *Let  $A = (L, \ell_0, F, C, \Sigma, I, E)$  be a timed automaton,  $R \subseteq L \times \mathbb{R}_{\geq 0}^C \times L \times \mathbb{R}_{\geq 0}^C$  an  $F$ -sensitive time-abstracted bisimulation and  $\ell \in F$ . Then  $\ell \in \text{Reach}(A)$  if and only if there is a reachable state  $\pi$  in  $\llbracket A \rrbracket_R$  and  $v : C \rightarrow \mathbb{R}_{\geq 0}$  such that  $(\ell, v) \in \pi$ .*

**Example 2.2 (continued).** We shall now try to construct, in a naïve way, a time-abstracted bisimulation  $R$  for the timed automaton  $A$  from Figure 2 which is as coarse as possible. Note first that we cannot have  $(\ell_0, v) R (\ell_1, v')$  for any  $v, v' : C \rightarrow \mathbb{R}_{\geq 0}$  because  $\ell_1 \in F$  and  $\ell_0 \notin F$ . On the other hand it is easy to see that we can let  $(\ell_1, v) R (\ell_1, v')$  for all  $v, v' : C \rightarrow \mathbb{R}_{\geq 0}$ , which leaves us with constructing  $R$  on the states involving  $\ell_0$ .

We handle switch transitions  $\xrightarrow{a}$  first: If  $v, v' : C \rightarrow \mathbb{R}_{\geq 0}$  are such that  $v(y) \leq 2$  and  $v'(y) > 2$ , the state  $(\ell_0, v)$  has an  $a$ -transition available while the state  $(\ell_0, v')$  has not, hence these cannot be related in  $R$ . Similarly we have to distinguish states  $(\ell_0, v)$  from states  $(\ell_0, v')$  where  $v(x) \leq 2$  and  $v'(x) > 2$  because of  $b$ -transitions, and states  $(\ell_0, v)$  from states  $(\ell_0, v')$  where  $v(x) < 4$  and  $v'(x) \geq 4$  because of  $c$ -transitions. Altogether this gives the five classes depicted to the left of Figure 3, where the shading indicates to which class the boundary belongs, and we have written the set of available actions in the classes.



**Figure 3.** Time-abstracted bisimulation classes for the two-clock timed automaton from Example 2.2. Left: equivalence classes for switch transitions only; right: equivalence classes for switch and delay transitions.

When also taking delay transitions  $\xrightarrow{\delta}$  into account, one has to partition the state space further: From a valuation  $v$  in the class marked  $\{a, b\}$  in the left of the figure, a valuation in the class marked  $\{a\}$  can only be reached by a delay transition if  $v(y) < v(x)$ ; likewise, from the  $\{a\}$  class, the  $\{a, c\}$  class can only be reached if  $v(y) \leq v(x) - 2$ . Hence these two classes need to be partitioned as shown to the right of Figure 3.

It can easily be shown that no further partitioning is needed, thus we have defined the coarsest time-abstracted bisimulation relation for  $A$ , altogether with eight equivalence classes.

### 2.3. Regions

Motivated by the construction in the example above, we now introduce a time-abstracted bisimulation with a *finite quotient*. To ensure finiteness, we need the maximal constants which respective clocks are compared to in the invariants and guards of a given timed automaton. These may be defined as below.

**Definition 2.14.** For a finite set  $C$  of clocks, the *maximal constant* mapping  $c_{\max} : C \rightarrow \mathbb{Z}^{\Phi(C)}$  is defined inductively as follows:

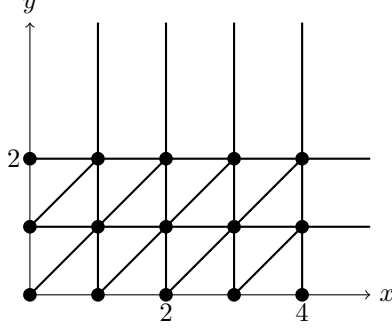
$$c_{\max}(x)(y \bowtie k) = \begin{cases} k & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases}$$

$$c_{\max}(x)(\varphi_1 \wedge \varphi_2) = \max(c(x)(\varphi_1), c(x)(\varphi_2))$$

For a timed automaton  $A = (L, \ell_0, F, C, \Sigma, I, E)$ , the maximal constant mapping is  $c_A : C \rightarrow \mathbb{Z}$  defined by

$$c_A(x) = \max \{ c_{\max}(x)(I(\ell)), c_{\max}(x)(\varphi) \mid \ell \in L, \ell \xrightarrow{\varphi, a, r} \ell' \in E \}$$

**Notation 2.15.** For  $d \in \mathbb{R}_{\geq 0}$  we write  $\lfloor d \rfloor$  and  $\langle d \rangle$  for the integral respectively fractional part of  $d$ , so that  $d = \lfloor d \rfloor + \langle d \rangle$ .



**Figure 4.** Clock regions for the timed automaton from Example 2.2.

**Definition 2.16.** For a timed automaton  $A = (L, \ell_0, F, C, \Sigma, I, E)$ , valuations  $v, v' : C \rightarrow \mathbb{R}_{\geq 0}$  are said to be *region equivalent*, denoted  $v \cong v'$ , if

- $\lfloor v(x) \rfloor = \lfloor v'(x) \rfloor$  or  $v(x), v'(x) > c_A(x)$ , for all  $x \in C$ , and
- $\langle v(x) \rangle = 0$  iff  $\langle v'(x) \rangle = 0$ , for all  $x \in C$ , and
- $\langle v(x) \rangle \leq \langle v(y) \rangle$  iff  $\langle v'(x) \rangle \leq \langle v'(y) \rangle$  for all  $x, y \in C$ .

**Proposition 2.17 ([4]).** For a timed automaton  $A = (L, \ell_0, F, C, \Sigma, I, E)$ , the relation  $\cong$  defined on states of  $\llbracket A \rrbracket$  by  $(\ell, v) \cong (\ell', v')$  if  $\ell = \ell'$  and  $v \cong v'$  is an  $F$ -sensitive time-abstracted bisimulation. The quotient  $\llbracket A \rrbracket_{\cong}$  is finite.

The equivalence classes of valuations of  $A$  with respect to  $\cong$  are called *regions*, and the quotient  $\llbracket A \rrbracket_{\cong}$  is called the *region automaton* associated with  $A$ .

**Proposition 2.18 ([8]).** The number of regions for a timed automaton  $A$  with a set  $C$  of  $n$  clocks is bounded above by

$$n! \cdot 2^n \cdot \prod_{x \in C} (2c_A(x) + 2)$$

**Example 2.2 (continued).** The 69 regions of the timed automaton  $A$  from Figure 2 are depicted in Figure 4.

Propositions 2.13 and 2.17 together now give the decidability part of the theorem below; for PSPACE-completeness see [6,37].

**Theorem 2.19.** The reachability problem for timed automata is PSPACE-complete.

#### 2.4. Behavioural refinement relations

We have already introduced time-abstracted simulations and bisimulations in Definition 2.11. As a corollary of Proposition 2.17, these are decidable:

**Theorem 2.20.** Time-abstracted simulation and bisimulation are decidable for timed automata.



The following provides a *time-sensitive* variant of (bi)simulation.

**Definition 2.21.** Let  $A = (L, \ell_0, F, C, \Sigma, I, E)$  be a timed automaton. A reflexive relation  $R \subseteq L \times \mathbb{R}_{\geq 0}^C \times L \times \mathbb{R}_{\geq 0}^C$  is a *timed simulation* provided that for all  $(\ell_1, v_1) R (\ell_2, v_2)$ ,

- for all  $(\ell_1, v_1) \xrightarrow{d} (\ell'_1, v'_1)$ ,  $d \in \mathbb{R}_{\geq 0}$ , there exists  $(\ell'_2, v'_2) R (\ell_1, v_1)$  for which  $(\ell_2, v_2) \xrightarrow{d} (\ell'_2, v'_2)$ , and
- for all  $(\ell_1, v_1) \xrightarrow{a} (\ell'_1, v'_1)$ ,  $a \in \Sigma$ , there exists  $(\ell'_2, v'_2) R (\ell_1, v_1)$  for which  $(\ell_2, v_2) \xrightarrow{a} (\ell'_2, v'_2)$ .

$R$  is called a *timed bisimulation* if it is also symmetric. Two states  $(\ell_1, v_1), (\ell_2, v_2) \in \llbracket A \rrbracket$  are said to be *timed bisimilar*, written  $(\ell_1, v_1) \sim (\ell_2, v_2)$ , if there exists a timed bisimulation  $R$  for which  $(\ell_1, v_1) R (\ell_2, v_2)$ .

Note that a timed (bi)simulation on  $A$  is the same as a standard (bi)simulation on  $\llbracket A \rrbracket$ .

**Definition 2.22.** Two timed automata  $A = (L^A, \ell_0^A, F^A, C^A, \Sigma^A, I^A, E^A)$  and  $B = (L^B, \ell_0^B, F^B, C^B, \Sigma^B, I^B, E^B)$  are said to be *timed bisimilar*, denoted  $A \sim B$ , if  $(\ell_0^A, v_0) \sim (\ell_0^B, v_0)$  in the disjoint-union transition system  $\llbracket A \rrbracket \sqcup \llbracket B \rrbracket$ .

Timed simulation of timed automata can be analogously defined. The following decidability result was established for *parallel timed processes* in [36]; below we give a version of the proof which has been adapted for timed automata.

**Theorem 2.23.** *Timed simulation and bisimulation are decidable for timed automata.*

Before the proof, we need a few auxiliary definitions and lemmas. The first is a product of timed transition systems which synchronizes on time, but not on actions:

**Definition 2.24.** The *independent product* of the timed transition systems  $\llbracket A \rrbracket = (S^A, s_0^A, \Sigma^A \cup \mathbb{R}_{\geq 0}, T^A)$ ,  $\llbracket B \rrbracket = (S^B, s_0^B, \Sigma^B \cup \mathbb{R}_{\geq 0}, T^B)$  associated with timed automata  $A, B$  is  $\llbracket A \rrbracket \times \llbracket B \rrbracket = (S, s_0, \Sigma^A \cup \Sigma^B \cup \mathbb{R}_{\geq 0}, T)$  given by

$$\begin{aligned} S &= S^A \times S^B & s_0 &= (s_0^A, s_0^B) \\ T &= \{(p, q) \xrightarrow{a} (p', q) \mid a \in \Sigma, p \xrightarrow{a} p' \in T^A\} \\ &\cup \{(p, q) \xrightarrow{b} (p, q') \mid b \in \Sigma, q \xrightarrow{b} q' \in T^B\} \\ &\cup \{(p, q) \xrightarrow{d} (p', q') \mid d \in \mathbb{R}_{\geq 0}, p \xrightarrow{d} p' \in T^A, q \xrightarrow{d} q' \in T^B\} \end{aligned}$$

We need to extend region equivalence  $\cong$  to the independent product. Below,  $\oplus$  denotes vector concatenation (direct sum); note that  $(p_1, q_1) \cong (p_2, q_2)$  is not the same as  $p_1 \cong q_1$  and  $p_2 \cong q_2$ , as fractional orderings  $\langle x^A \rangle \bowtie \langle x^B \rangle$ , for  $x^A \in C^A$ ,  $x^B \in C^B$ , have to be accounted for in the former, but not in the latter. Hence  $(p_1, q_1) \cong (p_2, q_2)$  implies  $p_1 \cong q_1$  and  $p_2 \cong q_2$ , but not vice-versa.

**Definition 2.25.** For states  $p_i = (\ell^{p_i}, v^{p_i})$  in  $\llbracket A \rrbracket$  and  $q_i = (\ell^{q_i}, v^{q_i})$  in  $\llbracket B \rrbracket$  for  $i = 1, 2$ , we say that  $(p_1, q_1) \cong (p_2, q_2)$  iff  $\ell^{p_1} = \ell^{p_2} \wedge \ell^{q_1} = \ell^{q_2}$  and  $v^{p_1} \oplus v^{q_1} \cong v^{p_2} \oplus v^{q_2}$ .

Note that the number of states in  $(\llbracket A \rrbracket \times \llbracket B \rrbracket)_{\cong}$  is finite, with an upper bound given by Proposition 2.18. Next we define transitions in  $(\llbracket A \rrbracket \times \llbracket B \rrbracket)_{\cong}$ :

**Notation 2.26.** Regions in  $(\llbracket A \rrbracket \times \llbracket B \rrbracket)_{\cong}$  will be denoted  $X, X'$ . The equivalence class of a pair  $(p, q) \in \llbracket A \rrbracket \times \llbracket B \rrbracket$  is denoted  $[p, q]$ .

**Definition 2.27.** For  $X, X' \in (\llbracket A \rrbracket \times \llbracket B \rrbracket)_{\cong}$  we say that

- $X \xrightarrow{a}_{\ell} X'$  for  $a \in \Sigma$  if for all  $(p, q) \in X$  there exists  $(p', q) \in X'$  such that  $(p, q) \xrightarrow{a} (p', q)$  in  $\llbracket A \rrbracket \times \llbracket B \rrbracket$ ,
- $X \xrightarrow{b}_{r} X'$  for  $b \in \Sigma$  if for all  $(p, q) \in X$  there exists  $(p, q') \in X'$  such that  $(p, q) \xrightarrow{b} (p, q')$  in  $\llbracket A \rrbracket \times \llbracket B \rrbracket$ , and
- $X \xrightarrow{\delta} X'$  if for all  $(p, q) \in X$  there exists  $d \in \mathbb{R}_{\geq 0}$  and  $(p', q') \in X'$  such that  $(p, q) \xrightarrow{d} (p', q')$ .

**Definition 2.28.** A subset  $\mathcal{B} \subseteq (\llbracket A \rrbracket \times \llbracket B \rrbracket)_{\cong}$  is a *symbolic bisimulation* provided that for all  $X \in \mathcal{B}$ ,

- whenever  $X \xrightarrow{a}_{\ell} X'$  for some  $X' \in (\llbracket A \rrbracket \times \llbracket B \rrbracket)_{\cong}$ , then  $X' \xrightarrow{a}_{r} X''$  for some  $X'' \in \mathcal{B}$ ,
- whenever  $X \xrightarrow{a}_{r} X'$  for some  $X' \in (\llbracket A \rrbracket \times \llbracket B \rrbracket)_{\cong}$ , then  $X' \xrightarrow{a}_{\ell} X''$  for some  $X'' \in \mathcal{B}$ , and
- whenever  $X \xrightarrow{\delta} X'$  for some  $X' \in (\llbracket A \rrbracket \times \llbracket B \rrbracket)_{\cong}$ , then  $X' \in \mathcal{B}$ .

Note that it is decidable whether  $(\llbracket A \rrbracket \times \llbracket B \rrbracket)_{\cong}$  admits a symbolic bisimulation. The following proposition finishes the proof of Theorem 2.23.

**Proposition 2.29.** *The quotient  $(\llbracket A \rrbracket \times \llbracket B \rrbracket)_{\cong}$  admits a symbolic bisimulation if and only if  $A \sim B$ .*

**Proof (cf. [36]):** For a given symbolic bisimulation  $\mathcal{B} \subseteq (\llbracket A \rrbracket \times \llbracket B \rrbracket)_{\cong}$ , the set  $R_{\mathcal{B}} = \{(p, q) \mid [p, q] \in \mathcal{B}\} \subseteq \llbracket A \rrbracket \times \llbracket B \rrbracket$  is a timed bisimulation. For the other direction, one can construct a symbolic bisimulation from a timed bisimulation  $R \subseteq \llbracket A \rrbracket \times \llbracket B \rrbracket$  by  $\mathcal{B}_R = \{[p, q] \mid (p, q) \in R\}$   $\square$

### 2.5. Language inclusion and equivalence

Similarly to the untimed setting, there is also a notion of language inclusion and equivalence for timed automata. We need to introduce the notion of *timed trace* first.

**Definition 2.30.** A *timed trace* over a finite set of actions  $\Sigma$  is a finite sequence  $((t_1, a_1), (t_2, a_2), \dots, (t_k, a_k))$ , where  $a_i \in \Sigma$  and  $t_i \in \mathbb{R}_{\geq 0}$  for  $i = 1, \dots, k$ , and  $t_i < t_{i+1}$  for  $i = 1, \dots, k-1$ . The set of all timed traces over  $\Sigma$  is denoted  $T\Sigma^*$ .

In a pair  $(t_i, a_i)$ , the number  $t_i$  is called the *time stamp* of the action  $a_i$ , *i.e.* the time at which event  $a_i$  occurs.

**Remark 2.31.** Timed traces as defined above are also known as *strongly monotonic* timed traces, because of the assumption that no consecutive events occur at the same time. *Weakly* monotonic timed traces, *i.e.* with requirement  $t_i \leq t_{i+1}$  instead of  $t_i < t_{i+1}$ , have also been considered, and there are some subtle differences between the two; see [65] for an important example.

**Definition 2.32.** A timed trace  $((t_1, a_1), \dots, (t_k, a_k))$  is *accepted* by a timed automaton  $A = (L, \ell_0, F, C, \Sigma, I, E)$  if there is an accepting run

$$\begin{aligned} (\ell_0, v_0) &\xrightarrow{t_1} (\ell_0, v_0 + t_1) \xrightarrow{a_1} (\ell_1, v_1) \xrightarrow{t_2 - t_1} \dots \\ &\dots \xrightarrow{a_{k-1}} (\ell_{k-1}, v_{k-1}) \xrightarrow{t_k - t_{k-1}} (\ell_{k-1}, v_{k-1} + t_k - t_{k-1}) \xrightarrow{a_k} (\ell_k, v_k) \end{aligned}$$

in  $A$ . The *timed language* of  $A$  is  $L(A) = \{\tau \in T\Sigma^* \mid \tau \text{ accepted by } A\}$ .

It is clear that  $L(A) = \emptyset$  if and only if none of the locations in  $F$  is reachable, hence Theorem 2.19 provides us with the decidability result in the following theorem. Undecidability of universality was established in [8]; we give an account of the proof below.

**Theorem 2.33.** *For a timed automaton  $A = (L, \ell_0, F, C, \Sigma, I, E)$ , deciding whether  $L(A) = \emptyset$  is PSPACE-complete. It is undecidable whether  $L(A) = T\Sigma^*$ .*

**Proof:** We may show that the universality problem for a timed automata is undecidable by reduction from the  $\Sigma_1^1$ -hard problem of deciding whether a given 2-counter machine  $M$  has a recurring computation.

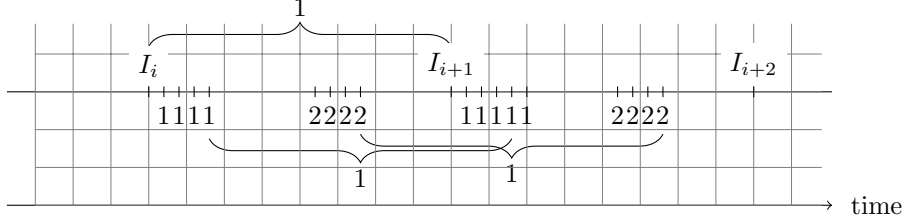
Let the timed language  $L_u$  be the set of timed traces encoding recurring computations of  $M$ . Observe that  $L_u = \emptyset$  if and only if  $M$  does not have such a computation. We then construct a timed automaton  $A_u$  which accepts the complement of  $L_u$ , *i.e.*  $L(A_u) = T\Sigma^* \setminus L_u$ . Hence the language of  $A_u$  is universal if and only if  $M$  does not have a recurring computation.

Recall that a 2-counter, or Minsky, machine  $M$  is a finite sequence of labeled instructions  $\{I_0, \dots, I_n\}$  and counters  $\mathbf{x}_1$  and  $\mathbf{x}_2$ , with  $I_i$  for  $0 \leq i \leq n-1$  on the form

$$I_i : \mathbf{x}_c := \mathbf{x}_c + 1; \text{ goto } I_j \quad \text{or} \quad I_i : \begin{cases} \text{if } \mathbf{x}_c = 0 \text{ then goto } I_j \\ \text{else } \mathbf{x}_c = \mathbf{x}_c - 1; \text{ goto } I_k \end{cases}$$

for  $c \in 1, 2$ , with a special  $I_n : \text{Halt}$  instruction which stops the computation.

The language  $L_u$  is designed such that each  $I_i$  and the counters  $\mathbf{x}_1$  and  $\mathbf{x}_2$  are represented by actions in  $\Sigma$ . A correctly encoded computation is represented by a timed trace where “instruction actions” occur at discrete intervals, while the state (values of  $\mathbf{x}_1$  and  $\mathbf{x}_2$ ) is encoded by occurrences of “counter actions” in-between instruction actions (*e.g.* if  $\mathbf{x}_i = 5$  after instruction  $I_j$ , then action  $x_i$  occurs 5 times within the succeeding interval of length 1).



**Figure 5.** Timed trace encoding an increment instruction  $I_{i+1}$  of a 2-counter machine.

When counters are incremented (or decremented), one more (or less) such action occurs through the next interval, and increments and decrements are always from the right. Additionally we require corresponding counter actions to occur exactly with a time difference of 1, such that if  $x_i$  occurs with time stamp  $a$  then also  $x_i$  occurs with time stamp  $a + 1$ , unless  $x_i$  is the rightmost  $x_i$  action and  $I_i$  at time stamp  $\lfloor a \rfloor$  is a decrement of  $x_i$ . Figure 5 shows an increment of  $x_1$  (from 4 to 5) using actions 1 and 2.

We obtain  $A_u$  as a disjunction of timed automata  $A^1, \dots, A^k$  where each  $A^i$  violates some property of a (correctly encoded) timed trace in  $L_u$ , either by accepting traces of incorrect format or inaccurate encodings of instructions.

Consider the instruction: (p):  $x_1 := x_1 + 1$  goto (q), incrementing  $x_1$  and jumping to q. A correct encoding would be similar to the one depicted in Figure 5 where all 1's and 2's are matched *one* time unit later, but with an additional 1 action occurring. In order to accept all traces except this encoding we must consider all possible violations, *i.e.*

- not incrementing the counter (no change),
- decrementing the counter,
- incrementing the counter more than once,
- jumping to the wrong instruction, or
- incrementing the wrong counter,

and construct a timed automaton having exactly such traces.

Figure 6 shows the timed automaton accepting traces in which instruction p yields no change of  $x_1$ . □

Turning our attention to timed trace inclusion and equivalence, we note the following.

**Proposition 2.34.** *Let  $A$  and  $B$  be timed automata. If  $A$  is timed simulated by  $B$ , then  $L(A) \subseteq L(B)$ . If  $A$  and  $B$  are timed bisimilar, then  $L(A) = L(B)$ .*

By a standard argument, Theorem 2.33 implies undecidability of timed trace inclusion and equivalence, a result first shown in [7].

**Theorem 2.35.** *Timed trace inclusion and equivalence are undecidable for timed automata.*

There is also a notion of *untimed* traces for timed automata.

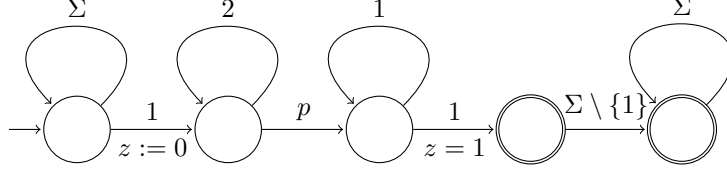


Figure 6. Timed automaton which violates the encoding of the increment instruction.

**Definition 2.36.** The *untiming* of a set of timed traces  $L \subseteq T\Sigma^*$  over a finite set of actions  $\Sigma$  is the set

$$UL = \{w = (a_1, \dots, a_k) \in \Sigma^* \mid \exists t_1, \dots, t_k \in \mathbb{R}_{\geq 0} : ((t_1, a_1), \dots, (t_k, a_k)) \in L\}$$

Hence we have a notion of the set  $UL(A)$  of *untimed language* of a timed automaton  $A$ . One can also define an untime operation  $U$  for timed automata, forgetting about the timing information of a timed automaton and thus converting it to a finite automaton; note however that  $UL(A) \subseteq L(UA)$  in general.

**Lemma 2.37 ([8]).** For  $A$  a timed automaton,  $UL(A) = L(\llbracket A \rrbracket_{\cong})$  provided that  $\delta$ -transitions in  $\llbracket A \rrbracket_{\cong}$  are taken as silent.

As a corollary, sets of untimed traces accepted by timed automata are *regular*:

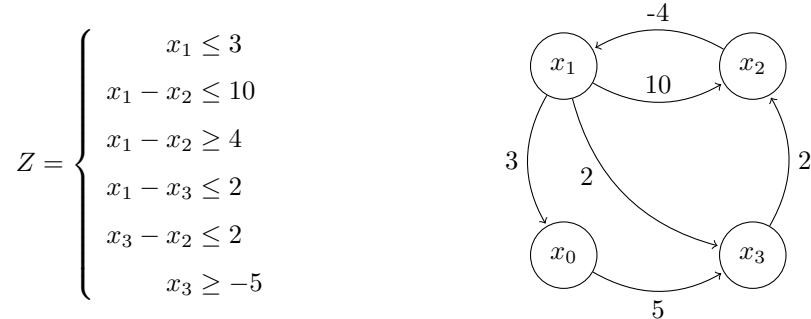
**Theorem 2.38.** For a timed automaton  $A = (L, \ell_0, F, C, \Sigma, I, E)$ , the set  $UL(A) \subseteq \Sigma^*$  is regular. Accordingly, whether  $UL(A) = \emptyset$  is decidable, and so is whether  $UL(A) = \Sigma^*$ . Also untimed trace inclusion and equivalence are decidable.

## 2.6. Zones and difference-bound matrices

As shown in the above sections, regions provide a finite and elegant abstraction of the infinite state space of timed automata, enabling us to prove decidability of reachability, timed and untimed bisimilarity, untimed language equivalence and language emptiness.

Unfortunately, the number of states obtained from the region partitioning is extremely large. In particular, by Proposition 2.18 the number of regions is exponential in the number of clocks as well as in the maximal constants of the timed automaton. Efforts have been made in developing more efficient representations of the state space [56,20,25], using the notion of *zones* from Definition 2.5 as a coarser and more compact representation of the state space.

An extended clock constraint over a finite set  $C$  may be represented using a directed weighted graph, where the nodes correspond to the elements of  $C$  together with an extra “zero” node  $x_0$ , and an edge  $x_i \xrightarrow{k} x_j$  corresponds to a constraint  $x_i - x_j \leq k$  (if there is more than one upper bound on  $x_i - x_j$ ,  $k$  is the minimum of all these constraints’ right-hand sides). The extra clock  $x_0$  is fixed at value 0, so that a constraint  $x_i \leq k$  can be represented as  $x_i - x_0 \leq k$ . Lower bounds on  $x_i - x_j$  are represented as (possibly negative) upper bounds on



**Figure 7.** Graph representation of extended clock constraint.

$x_j - x_i$ , and strict bounds  $x_i - x_j < k$  are represented by adding a flag to the corresponding edge.

The weighted graph in turn may be represented by its adjacency matrix, which is known as a *difference-bound matrix* or DBM. The above technique has been introduced in [39].

**Example 2.3.** Figure 7 gives an illustration of an extended clock constraint together with its representation as a difference-bound matrix. Note that the clock constraint contains superfluous information.

Zone-based reachability analysis of a timed automaton  $A$  uses symbolic states of the type  $(\ell, Z)$ , where  $\ell$  is a location of  $A$  and  $Z$  is a zone, instead of the region-based symbolic states of Proposition 2.17.

**Definition 2.39.** For a finite set  $C$ ,  $Z \subseteq \mathbb{R}_{\geq 0}^C$ , and  $r \subseteq C$ , define

- the *delay* of  $Z$  by  $Z^\dagger = \{v + d \mid v \in Z, d \in \mathbb{R}_{\geq 0}\}$  and
- the *reset* of  $Z$  under  $r$  by  $Z[r] = \{v[r] \mid v \in Z\}$ .

**Lemma 2.40 ([46,69]).** *If  $Z$  is a zone over  $C$  and  $r \subseteq C$ , then  $Z^\dagger$  and  $Z[r]$  are also zones over  $C$ .*

Extended clock constraints representing  $Z^\dagger$  and  $Z[r]$  may be computed efficiently (*i.e.* in time cubic in the number of clocks in  $C$ ) by representing the zone  $Z$  in a canonical form obtained by computing the *shortest-path closure* of the directed graph representation of  $Z$ , see [54].

**Example 2.3 (continued).** Figure 8 shows two canonical representations of the difference-bound matrix for the zone  $Z$  of Figure 7. The left part illustrates the shortest-path closure of  $Z$ ; on the right is the *shortest-path reduction* [54] of  $Z$ , essentially obtained by removing redundant edges from the shortest-path closure. The latter is useful for checking zone inclusion, see below.

The *zone automaton* associated with a timed automaton is similar to the region automaton of Proposition 2.17, but uses zones for symbolic states instead of regions:

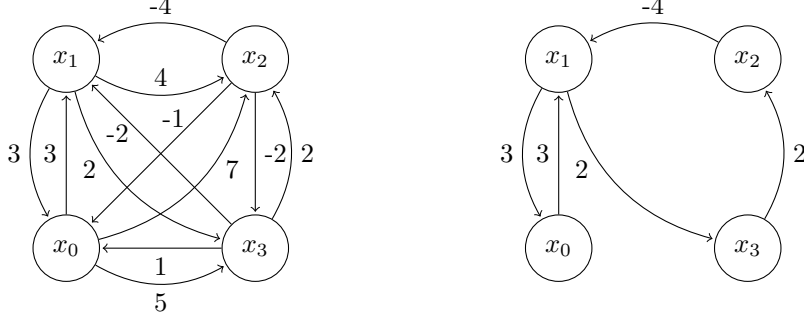


Figure 8. Canonical representations. Left: shortest-path closure; right: shortest-path reduction.

**Definition 2.41.** The *zone automaton* associated with a timed automaton  $A = (L, \ell_0, F, C, \Sigma, I, E)$  is the transition system  $\llbracket A \rrbracket_Z = (S, s_0, \Sigma \cup \{\delta\}, T)$  given by

$$\begin{aligned}
 S &= \{(\ell, Z) \mid \ell \in L, Z \subseteq \mathbb{R}_{\geq 0}^C \text{ zone}\} & s_0 &= (\ell_0, \llbracket v_0 \rrbracket) \\
 T &= \{(\ell, Z) \xrightarrow{\delta} (\ell, Z^\uparrow \wedge I(\ell))\} \\
 &\cup \{(\ell, Z) \xrightarrow{a} (\ell', (Z \wedge \varphi)[r] \wedge I(\ell')) \mid \ell \xrightarrow{\varphi, a, r} \ell' \in E\}
 \end{aligned}$$

The analogue of Proposition 2.13 for zone automata is as follows:

**Proposition 2.42 ([69]).** A state  $(\ell, v)$  in a timed automaton  $A = (L, \ell_0, F, C, \Sigma, I, E)$  is reachable if and only if there is a zone  $Z \subseteq \mathbb{R}_{\geq 0}^C$  for which  $v \in Z$  and such that  $(\ell, Z)$  is reachable in  $\llbracket A \rrbracket_Z$ .

The zone automaton associated with a given timed automaton is *infinite* and hence unsuitable for reachability analysis. Finiteness can be enforced by employing *normalization*, using the fact that region equivalence  $\cong$  has finitely many equivalence classes:

**Definition 2.43.** For a timed automaton  $A$  and a zone  $Z \subseteq \mathbb{R}_{\geq 0}^C$ , the *normalization* of  $Z$  is the set  $\{v : C \rightarrow \mathbb{R}_{\geq 0} \mid \exists v' \in D : v \cong v'\}$

The normalized zone automaton is defined in analogy to the zone automaton from above, and the analogue of Proposition 2.42 holds for the normalized zone automaton. Hence we can obtain a reachability algorithm by applying any search strategy (depth-first, breadth-first, or another) on the normalized zone automaton.

**Remark 2.44.** For timed automata on *extended* clock constraints, *i.e.* with diagonal constraints permitted, it can be shown [27,24] that normalization as defined above does *not* give rise to a sound and complete characterization of reachability. Instead, one can apply a refined normalization which depends on the difference constraints used in the timed automaton, see [24].

In addition to the efficient computation of symbolic successor states according to the  $\rightsquigarrow$  relation, termination of reachability analysis requires that we can

efficiently recognize whether the search algorithm has encountered a given symbolic state. Here it is crucial that there is an efficient way of deciding inclusion  $Z_1 \subseteq Z_2$  between zones. Both the shortest-path-closure canonical form as well as the more space-economical shortest-path-reduced canonical form [54], cf. Example 2.3, allow for efficient inclusion checking.

In analogy to difference-bound matrices and overcoming some of their problems, another data structure of *clock difference diagram* (CDD) has been proposed [56]. However, the design of efficient algorithms for delay and reset operations is a challenging open problem; generally, the design of efficient data structures for computations with (unions of) zones is a field of active research, see [3,64,11] for some examples.

### 3. Weighted timed automata

The notion of *weighted* — or *priced* — timed automata was introduced independently, at the very same conference, by Behrmann *et.al.* [18] and Alur *et.al.* [9]. In these models both edges and locations can be decorated with weights, or prices, giving the cost of taking an action transition or the cost per time unit of delaying in a given location. The total cost of a trace is then simply the accumulated (or total) weight of its discrete and delay transitions.

As a first result, the above two papers independently, and with quite different methods, showed that the problem of cost-optimal reachability is computable for weighted timed automata. Later, optimal reachability for timed automata with several weight functions was considered in [59] as well as optimal infinite runs in [41,29].

**Definition 3.1.** A *weighted timed automaton* is a tuple  $A = (L, \ell_0, F, C, \Sigma, I, E, R, P)$ , where  $(L, \ell_0, F, C, \Sigma, I, E)$  is a timed automaton,  $R : L \rightarrow \mathbb{Z}$  a location weight-rate mapping, and  $P : E \rightarrow \mathbb{Z}$  an edge weight mapping.

The *semantics* of  $A$  is the weighted transition system  $\llbracket A \rrbracket = (S, s_0, \Sigma \cup \mathbb{R}_{\geq 0}, T, w)$ , where  $(S, s_0, \Sigma \cup \mathbb{R}_{\geq 0}, T)$  is the semantics of the underlying timed automaton  $(L, \ell_0, F, C, \Sigma, I, E)$ , and the transition weights  $w : T \rightarrow \mathbb{R}$  are given as follows:

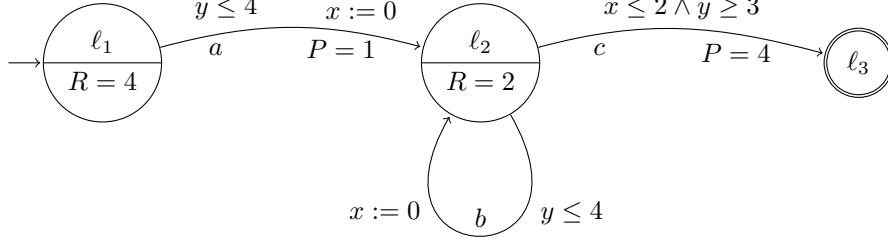
$$\begin{aligned} w((\ell, v) \xrightarrow{d} (\ell, v + d)) &= d R(\ell) \\ w((\ell, v) \xrightarrow{a} (\ell', v')) &= P(\ell \xrightarrow{\varphi, a, r} \ell') \quad \text{with } v \models \varphi, v' = v[r] \end{aligned}$$

We shall denote weighted edges and transitions by symbols  $\xrightarrow[e]{w}$  to illustrate an edge or a transition labeled  $e$  with weight  $w$ .

#### 3.1. Optimal reachability

The objective of optimal reachability analysis is to find runs to a final location with the lowest *total weight* as defined below.





**Figure 9.** A weighted timed automaton with two clocks.

**Example 3.1.** Figure 9 shows a simple weighted timed automaton with final location  $\ell_3$ . Below we give a few examples of accepting runs, where we identify valuations  $v : \{x, y\} \rightarrow \mathbb{R}_{\geq 0}$  with their values  $(v(x), v(y))$ . The total weights of the runs given here are 17 and 11.

$$\begin{aligned}
 (\ell_1, 0, 0) &\xrightarrow[12]{3} (\ell_1, 3, 3) \xrightarrow[1]{a} (\ell_2, 0, 3) \xrightarrow[4]{c} (\ell_3, 0, 3) \\
 (\ell_1, 0, 0) &\xrightarrow[1]{a} (\ell_2, 0, 0) \xrightarrow[6]{3} (\ell_2, 3, 3) \xrightarrow[0]{b} (\ell_2, 0, 3) \xrightarrow[4]{c} (\ell_3, 0, 3)
 \end{aligned}$$

**Definition 3.2.** The *total weight* of a finite run  $\rho = s_0 \xrightarrow[w_1]{s_1} \xrightarrow[w_2]{s_2} \cdots \xrightarrow[w_k]{s_k}$  in a weighted transition system is  $w(\rho) = \sum_{i=1}^k w_i$ .

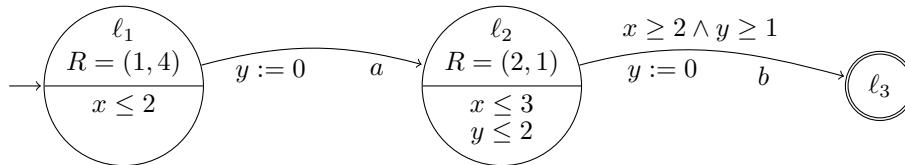
We are now in a position to state the problem with which we are concerned here: We want to find accepting runs with minimum total weight in a weighted timed automaton  $A$ . However due to the possible use of strict clock constraints on edges and in locations of  $A$ , the minimum total weight might not be realizable, *i.e.* there might be no run which achieves it. For this reason, one also needs to consider (infinite) *sets* of runs and the infimum of their members' total weights:

**Problem 3.1 (Optimal reachability).** Given a weighted timed automaton  $A$ , compute  $W = \inf \{w(\rho) \mid \rho \text{ accepting run in } A\}$  and a set  $P$  of accepting runs for which  $\inf_{\rho \in P} w(\rho) = W$ .

The key ingredient in the proof of the following theorem is the introduction of *weighted regions* in [18]. A weighted region is a region as of Definition 2.16 enriched with an affine cost function describing in a finite manner the cost of reaching any point within it. This notion allows to define the weighted region automaton associated with a weighted timed automaton, and one can then show that optimal reachability can be computed in the weighted region automaton.

**Theorem 3.3 ([18]).** *The optimal reachability problem is computable for weighted timed automata with non-negative weights.*

Similar to the notion of regions for timed automata, the number of weighted regions is exponential in the number of clocks as well as in the maximal constants of the timed automaton. Hence a notion of *weighted zone* — a zone extended with



**Figure 10.** A doubly weighted timed automaton with two clocks.

an affine cost function — was introduced [53] together with an efficient, symbolic  $A^*$ -algorithm for searching for cost-optimal tracing using branch-and-bound techniques. In particular, efficient means of generalizing the notion of symbolic successor to incorporate the affine cost functions were given.

During the symbolic exploration, several small linear-programming problems in terms of determining the minimal value of the cost function over the given zone have to be dealt with. Given that the constraints of these problems are simple difference constraints, it turns out that substantial gain in performance may be achieved by solving the dual problem of minimum-cost flow [67]. The newly emerged branch UPPAAL CORA provides an efficient tool for cost-optimal reachability analysis, applying the above data structures and algorithms and allowing the user to guide and heuristically prune the search.

### 3.2. Multi-weighted timed automata

**Definition 3.4.** An *doubly weighted timed automaton* is a tuple

$$A = (L, \ell_0, F, C, \Sigma, I, E, R, P)$$

where  $(L, \ell_0, F, C, \Sigma, I, E)$  is a timed automaton,  $R : L \rightarrow \mathbb{Z}^2$  a location weight-rate mapping, and  $P : E \rightarrow \mathbb{Z}^2$  an edge weight mapping.

The semantics of a doubly weighted timed automaton is a doubly weighted transition system defined similarly to Definition 3.1, and the total weight of finite paths is defined accordingly as a pair; we shall refer to the total weights as  $w_1$  and  $w_2$  respectively. These definitions have natural generalizations to *multi-weighted* timed automata with more than two weight coordinates.

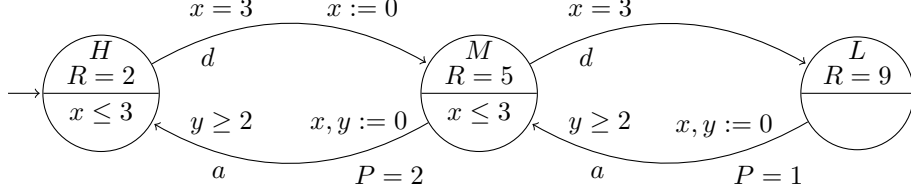
The objective of conditional reachability analysis is to find runs to a final location with the lowest total weight in the first weight coordinate while satisfying a constraint on the other weight coordinate.

**Example 3.2.** Figure 10 depicts a simple doubly weighted timed automaton with final location  $\ell_3$ . Under the constraint  $w_2 \leq 3$ , the optimal run of the automaton can be seen to be

$$(\ell_1, 0, 0) \xrightarrow{\frac{1/3}{(\frac{1}{3}, \frac{4}{3})}} (\ell_1, 1/3, 1/3) \xrightarrow{a} (\ell_2, 1/3, 0) \xrightarrow{\frac{5/3}{(\frac{10}{3}, \frac{5}{3})}} (\ell_2, 2, 5/3) \xrightarrow{b} (\ell_3, 2, 0)$$

with total weight  $(\frac{11}{3}, 3)$ .

The precise formulation of the conditional optimal reachability problem is as follows, where we again need to refer to (possibly infinite) sets of runs:



**Figure 11.** A weighted timed automaton modelling a simple production system.

**Problem 3.2 (Conditional optimal reachability).** Given a doubly weighted timed automaton  $A$  and  $M \in \mathbb{Z}$ , compute  $W = \inf \{w_1(\rho) \mid \rho \text{ accepting run in } A, w_2(\rho) \leq M\}$  and a set  $P$  of accepting runs such that  $w_2(\rho) \leq M$  for all  $\rho \in P$  and  $\inf_{\rho \in P} w(\rho) = W$ .

**Theorem 3.5 ([58,59]).** *The conditional optimal reachability problem is computable for doubly weighted timed automata with non-negative weights and without weights on edges.*

The proof of the above theorem rests on a direct generalization of weighted to *doubly-weighted zones*. An extension can be found in [59], where it is shown that also the *Pareto frontier*, *i.e.* the set of cost vectors which cannot be improved in any cost variable, can be computed.

### 3.3. Optimal infinite runs

In this section we shall be concerned with computing optimal *infinite runs* in (doubly) weighted timed automata. We shall treat both the *limit ratio* viewpoint discussed in [29] and the *discounting* approach of [41].

**Example 3.3.** Figure 11 shows a simple production system modelled as a weighted timed automaton. The system has three modes of production, High, Medium, and Low. The weights model the *cost* of production, so that the High production mode has a low cost, which is preferable to the high cost of the Low production mode. After operating in a High or Medium production mode for three time units, production automatically degrades (action  $d$ ) to a lower mode. When in Medium or Low production mode, the system can be attended to (action  $a$ ), which advances it to a higher mode.

The objective of *optimal-ratio analysis* is to find an infinite run in a doubly weighted timed automaton which minimizes the *ratio* between the two total weights. This will be formalized below.

**Definition 3.6.** The *total ratio* of a finite path  $\rho = s_0 \xrightarrow[z_1]{w_1} s_1 \xrightarrow[z_2]{w_2} \dots \xrightarrow[z_k]{w_k} s_k$  in a doubly weighted transition system is

$$\Gamma(\rho) = \frac{\sum_{i=1}^k w_i}{\sum_{i=1}^k z_i}$$

The total ratio of an infinite path  $\rho = s_0 \xrightarrow[z_1]{w_1} s_1 \xrightarrow[z_2]{w_2} \dots$  is

$$\Gamma(\rho) = \liminf_{k \rightarrow \infty} \Gamma(s_0 \rightarrow \dots \rightarrow s_k)$$

A special case of optimal-ratio analysis is given by weight-per-time models, where the interest is in minimizing total weight per accumulated time. The example provided in this section is a case of this. In the setting of optimal-ratio analysis, these can be modelled as doubly weighted timed automata with  $R_2(\ell) = 1$  and  $P_2(e) = 0$  for all locations  $\ell$  and edges  $e$ .

**Example 3.3 (continued).** In the timed automaton of Figure 11, the following cyclic behaviour provides an infinite run  $\rho$ :

$$\begin{aligned} (H, 0, 0) \xrightarrow{3} (H, 3, 3) \xrightarrow{d} (M, 0, 3) \xrightarrow{3} (M, 3, 6) \xrightarrow{d} (L, 3, 6) \xrightarrow{1} \\ (L, 4, 7) \xrightarrow{a} (M, 0, 0) \xrightarrow{3} (M, 3, 3) \xrightarrow{a} (H, 0, 0) \rightarrow \dots \end{aligned}$$

Taking the weight-per-time viewpoint, the total ratio of  $\rho$  is  $\Gamma(\rho) = 4.8$ .

**Problem 3.3 (Minimum infinite ratio).** Given a doubly weighted timed automaton  $A$ , compute  $W = \inf \{ \Gamma(\rho) \mid \pi \text{ infinite run in } A \}$  and a set  $P$  of infinite runs for which  $\inf_{\pi \in P} \Gamma(\pi) = W$ . *llllllll .r180*

The main tool in the proof of the following theorem is the introduction of the *corner-point abstraction* of a timed automaton in [29]. This is a finite refinement of the region automaton of Definition 2.16 in which one also keeps track of the corner points of regions. One can then show that any infinite run with minimum ratio must pass through corner points of regions, hence these can be found in the corner-point abstraction by an algorithm first proposed in [51].

The technical condition in the theorem that the second weight coordinate be *strongly diverging* means that any infinite run  $\rho$  in the closure of the timed automaton in question satisfy  $w_2(\rho) = \infty$ , see [29] for details.

**Theorem 3.7 ([29]).** *The minimum infinite ratio problem is computable for doubly weighted timed automata with non-negative and strongly diverging second weight coordinate.*

For *discount-optimal analysis*, the objective is to find an infinite run in a weighted timed automaton which minimizes the *discounted total weight* as defined below. The point of discounting is that the weight of actions is discounted with time, so that the impact of an event decreases, the further in the future it takes place.

In the definition below,  $\varepsilon$  is the empty run, and  $(\ell, v) \rightarrow \pi$  denotes the concatenation of the transition  $(\ell, v) \rightarrow$  with the run  $\pi$ .

**Definition 3.8.** The *discounted total weight* of finite runs in a weighted timed automaton under discounting factor  $\lambda \in [0, 1[$  is given inductively by

$$\begin{aligned}
w_\lambda(\varepsilon) &= 0 \\
w_\lambda((\ell, v) \xrightarrow{a} \rho) &= P + w_\lambda(\rho) \\
w_\lambda((\ell, v) \xrightarrow{d} \rho) &= R(\ell) \int_0^d \lambda^\tau d\tau + \lambda^d w_\lambda(\rho)
\end{aligned}$$

The discounted total weight of an infinite run  $\rho = (\ell_0, v_0) \xrightarrow{a_1} (\ell_0, v_0 + d_1) \xrightarrow{a_1} (\ell_1, v_1) \rightarrow \dots$  is

$$w_\lambda(\rho) = \lim_{k \rightarrow \infty} w_\lambda((\ell_0, v_0) \rightarrow \dots \xrightarrow{a_k} (\ell_k, v_k))$$

provided that the limit exists.

**Example 3.3 (continued).** The discounted total weight of the infinite run  $\rho$  in the timed automaton of Figure 11 satisfies the following equality, where  $I_t = \int_0^t \lambda^\tau d\tau = -\frac{1}{\ln \lambda}(1 - \lambda^t)$ :

$$w_\lambda(\rho) = 2I_3 + \lambda^3(5I_3 + \lambda^3(9I_1 + \lambda(1 + 5I_3 + \lambda^3(2 + w_\lambda(\rho))))))$$

With a discounting factor of  $\lambda = .9$  for example, the discounted total weight of  $\rho$  would hence be  $w_\lambda(\rho) \approx 40.5$ .

**Problem 3.4 (Minimum discounted weight).** Given a weighted timed automaton  $A$  and  $\lambda \in [0, 1[$ , compute  $W = \inf \{w_\lambda(\rho) \mid \rho \text{ infinite run in } A\}$  and a set  $P$  of infinite runs for which  $\inf_{\pi \in P} w_\lambda(\pi) = W$ . *lllllll .r180*

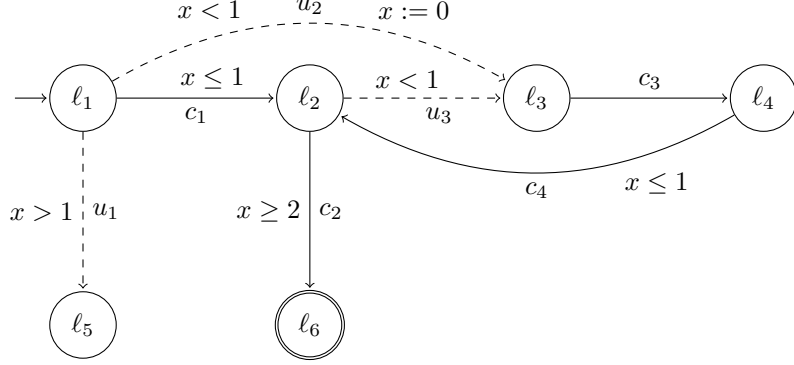
The proof of the following theorem rests again on the corner-point abstraction, and on a result in [10]. The technical condition that the timed automaton be time-divergent is analogous to the condition on the second weight coordinate in Theorem 3.7.

**Theorem 3.9 ([41]).** *The minimum discounted weight problem is computable for time-divergent weighted timed automata with non-negative weights and rational  $\lambda$ .*

#### 4. Timed games

Recently, substantial effort has been made towards the synthesis of winning strategies for timed games with respect to *safety* and *reachability control objectives*. From known region-based decidability results, efficient on-the-fly algorithms have been developed [68,34] and implemented in the newest branch UPPAAL TIGA.

For timed games, as for untimed ones, transitions are either controllable or uncontrollable (*i.e.* under the control of an environment), and the problem is to synthesize a strategy for *when* to take *which* (enabled) controllable transitions in order that a given objective is guaranteed regardless of the behaviour of the environment.



**Figure 12.** A timed game with one clock. Controllable edges (with actions from  $\Sigma_c$ ) are solid, uncontrollable edges (with actions from  $\Sigma_u$ ) are dashed.

**Definition 4.1.** A *timed game* is a tuple  $(L, \ell_0, F, C, \Sigma_c, \Sigma_u, I, E)$  for which the tuple  $(L, \ell_0, F, C, \Sigma = \Sigma_c \cup \Sigma_u, I, E)$  is a timed automaton.

Edges with actions in  $\Sigma_c$  are said to be *controllable*, those with actions in  $\Sigma_u$  are *uncontrollable*.

**Example 4.1.** Figure 12 provides an example of a timed game. Here,  $\Sigma_c = \{c_1, c_2, c_4\}$  and  $\Sigma_u = \{u_1, u_2, u_3\}$ , and the controllable edges are drawn with solid lines, the uncontrollable ones with dashed lines.

We only treat *reachability* games here, where the goal of the game is to reach a final location. There is also a somewhat dual notion of *safety* games, where one instead wants to *avoid* final locations, see [34] for details.

We need the notion of *strategy*; essentially, a strategy provides instructions for which controllable edge to take, or whether to wait, in a given state:

**Definition 4.2.** A *strategy* for a timed game  $A = (L, \ell_0, F, C, \Sigma_c, \Sigma_u, I, E)$  is a partial mapping  $\sigma$  from finite runs of  $A$  to  $\Sigma_c \cup \{\delta\}$ , where  $\delta \notin \Sigma$ , such that for any run  $\rho = (\ell_0, v_0) \rightarrow \dots \rightarrow (\ell_k, v_k)$ ,

- if  $\sigma(\rho) = \delta$ , then  $(\ell, v) \xrightarrow{d} (\ell, v + d)$  in  $\llbracket A \rrbracket$  for some  $d > 0$ , and
- if  $\sigma(\rho) = a$ , then  $(\ell, v) \xrightarrow{a} (\ell', v')$  in  $\llbracket A \rrbracket$ .

A strategy  $\sigma$  is said to be *memoryless* if  $\sigma(\rho)$  only depends on the last state of  $\rho$ , i.e. if  $\rho_1 = (\ell_0, v_0) \xrightarrow{d_1} (\ell_0, v_0 + d_1) \rightarrow \dots \rightarrow (\ell_k, v_k)$ ,  $\rho_2 = (\ell_0, v_0) \xrightarrow{d'_1} (\ell_0, v_0 + d'_1) \rightarrow \dots \rightarrow (\ell_k, v_k)$  imply  $\sigma(\rho_1) = \sigma(\rho_2)$ .

An *outcome* of a strategy is any run which adheres to its instructions in the obvious manner:

**Definition 4.3.** A run  $(\ell_0, v_0) \xrightarrow{d_1} (\ell_0, v_0 + d_1) \rightarrow \dots \rightarrow (\ell_k, v_k)$  in a timed game  $A = (L, \ell_0, F, C, \Sigma_c, \Sigma_u, I, E)$  is said to be an *outcome* of a strategy  $\sigma$  provided that

- for all  $(\ell_i, v_i) \xrightarrow{d} (\ell_i, v_i + d)$  and for all  $d' < d$ , we have  $\sigma((\ell_0, v_0) \rightarrow \dots \rightarrow (\ell_i, v_i + d')) = \delta$ , and
- for all  $(\ell_i, v_i + d) \xrightarrow{a} (\ell_{i+1}, v_{i+1})$  for which  $a \in \Sigma_c$ , we have  $\sigma((\ell_0, v_0) \rightarrow \dots \rightarrow (\ell_i, v_i)) = a$ .

An outcome is said to be *maximal* if  $\ell_k \in F$ , or if  $(\ell_k, v_k) \xrightarrow{a} (\ell_{k+1}, v_{k+1})$  implies  $a \in \Sigma_u$ .

Hence an outcome is maximal if it stops in a final state, or if no controllable actions are available at its end. An underlying assumption is that uncontrollable actions cannot be forced, hence a maximal outcome which does not end in a final state may “get stuck” in a non-final state. The aim of reachability games is to find strategies all of whose maximal outcomes end in a final state:

**Definition 4.4.** A strategy is said to be *winning* if any of its maximal outcomes is an accepting run.

**Example 4.1 (continued).** The following memoryless strategy is winning for the reachability game on the timed game from Figure 12:

$$\begin{aligned} \sigma(\ell_1, v) &= \begin{cases} \delta & \text{if } v(x) \neq 1 \\ c_1 & \text{if } v(x) = 1 \end{cases} & \sigma(\ell_2, v) &= \begin{cases} \delta & \text{if } v(x) < 2 \\ c_2 & \text{if } v(x) \geq 2 \end{cases} \\ \sigma(\ell_3, v) &= \begin{cases} \delta & \text{if } v(x) < 1 \\ c_3 & \text{if } v(x) \geq 1 \end{cases} & \sigma(\ell_4, v) &= \begin{cases} \delta & \text{if } v(x) \neq 1 \\ c_4 & \text{if } x(x) = 1 \end{cases} \end{aligned}$$

**Problem 4.1 (Reachability game).** Given a timed game  $A$ , does there exist a winning strategy for  $A$ ?

An important ingredient in the proof of the following theorem is the fact that for reachability (as well as safety) games, it is sufficient to consider *memoryless* strategies. This is not the case for other, more subtle, control objectives (e.g. counting properties modulo some  $N$ ) as well as for the synthesis of winning strategies under *partial observability*.

**Theorem 4.5 ([63,12]).** *The reachability game is decidable for timed games.*

In [35] the on-the-fly algorithm applied in UPPAAL TIGA has been extended to timed games under partial observability.

The field of timed games is a very active research area. Research has been conducted towards the synthesis of *optimal* winning strategies for reachability games on *weighted timed games*. In [5,30] computability of optimal strategies is shown under a certain condition of *strong cost non-zenoness*, requiring that the total weight diverges with a given minimum rate per time. Later undecidability results [33,28] show that for weighted timed games with three or more clocks this condition (or a similar one) is necessary. Lately [31] proves that optimal reachability strategies are computable for one-clock weighted timed games, though there is an unsettled (large) gap between the known lower bound complexity P and an upper bound of 3EXPTIME.

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