An Algebraic Theory of Markov Processes

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LICS'18
9th July 2018, Oxford
Historical Perspective

- **Moggi'88**: How to incorporate effects into denotational semantics? - **Monads** as notions of computations

- **Plotkin & Power'01**: (most of the) Monads are given by operations and equations - **Algebraic Effects**

- **Hyland, Plotkin, Power'06**: sum and tensor of theories - Combining Algebraic Effects

- **Mardare, Panangaden, Plotkin (LICS'16)**: Theory of effects in a metric setting - **Quantitative Algebraic Effects** (operations & quantitative equations give monads on Met)

\[ s = \epsilon t \]
Quantitative Equations

\[ S = \varepsilon t \]

"\( S \) is approximately equal to \( t \) up to an error \( \varepsilon \)"
What have we done

• Shown how to combine -by disjoint union- different theories to produce new interesting examples

• Specifically, equational axiomatization of Markov processes obtained by combining equations for transition systems and equations for probability distributions

• The equations are in the generalized quantitative sense of Mardare et al. LICS'16

• We have characterized the final coalgebra of Markov processes algebraically
Quantitative Equational Theory

Mardare, Panangaden, Plotkin (LICS’16)

A quantitative equational theory $\mathcal{U}$ of type $\Sigma$ is a set of

$$\left\{ t_i = \epsilon_i \ s_i \mid i \in I \right\} \vdash t = \epsilon s$$

closed under the following "meta axioms"

(Refl) $\vdash x =_0 x$

(Symm) $x =_\epsilon y \vdash y =_\epsilon x$

(Triang) $x =_\epsilon y, y =_\delta z \vdash x =_{\epsilon+\delta} y$

(NExp) $x_1 =_\epsilon y_1, \ldots, y_n =_\epsilon y_n \vdash f(x_1, \ldots, x_n) =_\epsilon f(y_1, \ldots, y_n) - \text{for } f \in \Sigma$

(Max) $x =_\epsilon y \vdash x =_{\epsilon+\delta} y - \text{for } \delta > 0$

(Inf) $\left\{ x =_\epsilon y \mid \delta > \epsilon \right\} \vdash x =_\epsilon y$

(1-Bdd*) $\vdash x =_1 y$
Quantitative Algebras
Mardare, Panangaden, Plotkin (LICS’16)

The models of a quantitative equational theory $\mathcal{U}$ of type $\Sigma$ are

**Quantitative $\Sigma$-Algebras:**

$\mathcal{A} = (A, \alpha : \Sigma A \to A)$ — **Universal $\Sigma$-algebras on Met**

Satisfying the all the quantitative equations in $\mathcal{U}$

We denote the category of models of $\mathcal{U}$ by

$\mathbb{K}(\Sigma, \mathcal{U})$
Standard picture

Monads on $\text{Set}$

Operations & Equations

EM category $\cong$ Algebras
Our picture

Monads on \( \text{Met} \)

Operations & Quantitative Equations

EM category \( \cong \) Quantitative Algebras
\( \mathcal{U} \) Models are \( T_{\mathcal{U}} \)-Algebras

\[
\{ x_i = \varepsilon_i y_i \mid i \in I \} \vdash t = \varepsilon s
\]

A quantitative equational theory \( \mathcal{U} \) is \textit{basic} if it can be axiomatised by a set of basic conditional quantitative equations.

\[ \textbf{Theorem} \]

For any \textit{basic} quantitative equational theory \( \mathcal{U} \) of type \( \Sigma \)

\[ \mathbb{K}(\Sigma, \mathcal{U}) \cong T_{\mathcal{U}}\text{-Alg} \]

\textbf{EM} algebras for the monad \( T_{\mathcal{U}} \)
Free Monads on CMet

A quantitative equational theory is continuous if it can be axiomatised by a collection of continuous schemata of quantitative equations

\[ x_1 = \varepsilon_1 y_1, \ldots, x_n = \varepsilon_n y_n \vdash t = \varepsilon s \quad \text{for } \varepsilon \geq f(\varepsilon_1, \ldots, \varepsilon_n) \]

\[ \mathbb{K}(\Sigma, \mathcal{U}) \xrightarrow{\hat{C}} \mathbb{C}\mathbb{K}(\Sigma, \mathcal{U}) \]

Models of \( \mathcal{U} \) over complete metric spaces

\[ \mathbb{C} \]

\[ \mathbb{K}(\Sigma, \mathcal{U}) \xrightarrow{\hat{C}} \mathbb{C}\mathbb{K}(\Sigma, \mathcal{U}) \]

\[ \mathbb{C} \]

\[ \mathbb{C} \]

\[ \mathbb{C} \]

\[ T_{\mathcal{U}} \]

\[ \mathbb{C}T_{\mathcal{U}} \]
Theory of Contractive Operators

The theory \( \mathcal{O}(\Sigma) \) induced by the axioms above is called \textit{quantitative equational theory of contractive operators over} \( \Sigma \)

\( f: \langle n, c \rangle \in \Sigma \)

\[(f\text{-Lip}) \{ x_1 =_\varepsilon y_1, \ldots, y_n =_\varepsilon y_n \} \vdash f(x_1, \ldots, x_n) =_\delta f(y_1, \ldots, y_n) - \text{for } \delta \geq c\varepsilon \]

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The theory of contractive operators

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The theory \( \mathcal{O}(\Sigma) \) induced by the axioms above is called quantitative equational theory of contractive operators over \( \Sigma \).

Monads

\[
T_{\mathcal{O}(\Sigma)} \cong \tilde{\Sigma}^*
\]
(on Met)

\[
\mathcal{C}T_{\mathcal{O}(\Sigma)} \cong \tilde{\Sigma}^*
\]
(on CSMet)
Interpolative Barycentric Theory

\[ \Sigma_{\mathcal{B}} = \{ +_e : 2 \mid e \in [0,1] \} \]

(B1) \( \vdash x +_1 y =_0 x \)

(B2) \( \vdash x +_e x =_0 x \)

(SC) \( \vdash x +_e y =_0 y +_1 -_e x \)

(SA) \( \vdash (x +_e y) +_d z =_0 x +_e d (y + (1 - ed) z) \quad \text{for} \quad e, d \in [0,1) \)

(IB) \( x =_e y, x' =_e' y' \vdash x +_e x' =_\delta y +_e y' \quad \text{for} \quad \delta \geq ee + (1 - e)e' \)

The quantitative theory \( \mathcal{B} \) induced by the axioms above is called

*interpolative barycentric quantitative equational theory*
Interpolative Barycentric Theory

Mardare, Panangaden, Plotkin (LICS’16)

\[ \Sigma \mathcal{B} = \{ +_e : 2 \mid e \in [0,1] \} \]

\( \text{(B1)} \quad \vdash x +_1 y =_0 x \)

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\( \text{(SC)} \quad \vdash x +_e y =_0 y +_{1-e} x \)

\( \text{(SA)} \quad \vdash (x +_e y) +_d z =_0 x +_e (y +_{d-\frac{ed}{1-ed}} z) \quad \text{for} \ e, d \in [0,1) \)

\( \text{(IB)} \quad x =_e y, x' =_{e'} y' \quad \vdash x +_e x' =_{\delta} y +_e y' \quad \text{for} \ \delta \geq e\epsilon + (1 - e)e' \)

The quantitative theory \( \mathcal{B} \) induced by the axioms above is called \textit{interpolative barycentric quantitative equational theory}.

\[ T_{\mathcal{B}} \cong \Pi \]

\text{(on Met)}

\[ C T_{\mathcal{B}} \cong \Delta \]

\text{(on CSMet)}

Finitely supported Borel probability measures with \textit{Kantorovich metric}

\textbf{Monads}

\textit{Borel probability measures with Kantorovich metric (Giry Monad)}
Disjoint Union of Theories

The disjoint union \( \mathcal{U} + \mathcal{U}' \) of two quantitative theories with disjoint signatures is the smallest quantitative theory containing \( \mathcal{U} \) and \( \mathcal{U}' \).

\[
\begin{align*}
\mathcal{K}(\Sigma, \mathcal{U}) & \vdash \text{Met} \downarrow T_\mathcal{U} \\
\mathcal{K}(\Sigma', \mathcal{U}') & \vdash \text{Met} \downarrow T_\mathcal{U}' \\
\mathcal{K}(\Sigma + \Sigma', \mathcal{U} + \mathcal{U}') & \vdash \text{Met} \downarrow T_{\mathcal{U}+\mathcal{U}'}
\end{align*}
\]

Models of \( \mathcal{U} + \mathcal{U}' \)
Disjoint Union of Theories

The disjoint union $\mathcal{U} + \mathcal{U}'$ of two quantitative theories with disjoint signatures is the smallest quantitative theory containing $\mathcal{U}$ and $\mathcal{U}'$.

\[ \mathcal{K}(\Sigma, \mathcal{U}) \quad \vdash \quad \mathcal{K}(\Sigma', \mathcal{U}') \quad \vdash \quad \mathcal{K}(\Sigma + \Sigma', \mathcal{U} + \mathcal{U}') \]

\[ \text{Met} \quad \downarrow \quad T_{\mathcal{U}} \quad + \quad T_{\mathcal{U}'} \quad \cong \quad \downarrow \quad T_{\mathcal{U} + \mathcal{U}'} \]

Models of $\mathcal{U} + \mathcal{U}'$
Disjoint Union of Theories

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Disjoint Union of Theories

The answer is positive for \textit{basic} quantitative theories

\[
T_\mathcal{U} + T_\mathcal{U}' \simeq T_{\mathcal{U}+\mathcal{U}}
\]

The proof follows standard techniques (Kelly'80)

\textbf{Theorem}

For \textit{basic} quantitative equational theories \(\mathcal{U}, \mathcal{U}'\) of type \(\Sigma, \Sigma'\)

\[
\mathbb{K}(\Sigma + \Sigma', \mathcal{U} + \mathcal{U}') \simeq \langle T_\mathcal{U}, T_\mathcal{U}' \rangle-\text{Alg} \simeq (T_\mathcal{U} + T_\mathcal{U}')-\text{Alg}
\]

\textit{EM-bialgebras} for the monads \(T_\mathcal{U}, T_\mathcal{U}'\)
Interpolative Barycentric Theory with Contractive Operators

\[ \Sigma_B + \Sigma = \{ +_e : 2 \mid e \in [0,1] \} \cup \Sigma \]

**B**

- (B1) \( \vdash x +_1 y = 0 x \)
- (B2) \( \vdash x +_e x = 0 x \)
- (SC) \( \vdash x +_e y = 0 y +_{1-e} x \)
- (SA) \( \vdash (x +_e y) +_d z = 0 x +_{ed} (y +_{d-ed} z) \) for \( e, d \in [0,1) \)
- (IB) \( x =_e y, x' =_e y' \vdash x +_e x' =_\delta y +_e y' \) for \( \delta \geq e\varepsilon + (1 - e)\varepsilon' \)

**\( \mathcal{O}(\Sigma) \)**

- (f-Lip) \( x_1 =_e y_1, \ldots, y_n =_e y_n \vdash f(x_1, \ldots, x_n) =_\delta f(y_1, \ldots, y_n) \) for \( \delta \geq c\varepsilon \)

Monads

\[ T_{B+\mathcal{O}(\Sigma)} \cong \Pi + \tilde{\Sigma}^* \] (on Met)
\[ \mathcal{C}T_{B+\mathcal{O}(\Sigma)} \cong \Delta + \tilde{\Sigma}^* \] (on CSMet)
Sum with Free Monad

Hyland, Plotkin, Power (TCS 2016)

**Theorem**

For a functor $F$ and a monad $T$, if the free monads $F^*$ and $(FT)^*$ exist, then the sum of monads $T + F^*$ exists and is given by a canonical monad structure on the composite $T(FT)^*$

**Corollary**

Under same assumptions as above, the sum of monads $T + F^*$ is given by a canonical monad structure on $\mu y. T(Fy + -$)
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generalised resumption monad of (Cenciarelli, Moggi'93)
Markov Process Monads

We can recover a quantitative theory of Markov Processes as an interpolative barycentric theory with the following signature of operators

\[ M_c = \{ 0: \langle 0, c \rangle, \diamond : \langle 1, c \rangle \} \quad (\text{for } 0 < c < 1) \]
Markov Process Monads

We can recover a quantitative theory of Markov Processes as an interpolative barycentric theory with the following signature of operators

\[ \mathcal{M}_c = \{ 0: \langle 0, c \rangle, \diamond : \langle 1, c \rangle \} \quad (\text{for } 0 < c < 1) \]

**Monads**

- **Rooted acyclic finite Markov processes, with c-probabilistic bisimilarity metric**

\[ T_{\mathcal{B} + \mathcal{O}}(\mathcal{M}_c) \cong \mu y \cdot \Pi(1 + c \cdot y + -) \]

- **Markov processes on complete separable metric spaces with c-probabilistic bisimilarity metric**

\[ \mathcal{T}_{\mathcal{B} + \mathcal{O}}(\mathcal{M}_c) \cong \mu y \cdot \Delta(1 + c \cdot y + -) \]
Final Coalgebra of MPs

\[ \mathbb{C} T_{\mathcal{B} + \mathcal{O}(\mathcal{M}_c)} \cong \mu y \cdot \Delta(1 + c \cdot y + - ) \]

assigns to any \( A \in \text{CSMet} \) the initial solution of the equation

\[ MP_A \cong \Delta(1 + c \cdot MP_A + A) \]
Final Coalgebra of MPs

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**Theorem (Turi, Rutten'98)**

Every *locally contractive functor* \( H \) on \( \text{CMet} \) has a unique fixed point, which is both an *initial algebra* and a *final coalgebra for* \( H \)
Final Coalgebra of MPs

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Theorem (Turi, Rutten'98)

Every locally contractive functor \( H \) on \( \text{CMet} \) has a unique fixed point, which is both an initial algebra and a final coalgebra for \( H \).

In particular, when \( A \in \emptyset \) (the empty metric space)

\[ MP_0 \rightarrow \Delta(1 + c \cdot MP_0) \]

final coalgebra of Markov processes
Conclusions

• Sum of quantitative theories (this opens the way to developing combinations of quantitative effects)

• Unifying algebraic and coalgebraic presentation of Markov processes (coincidence with initial and final coalgebra)

• Tensor product of quantitative theories?