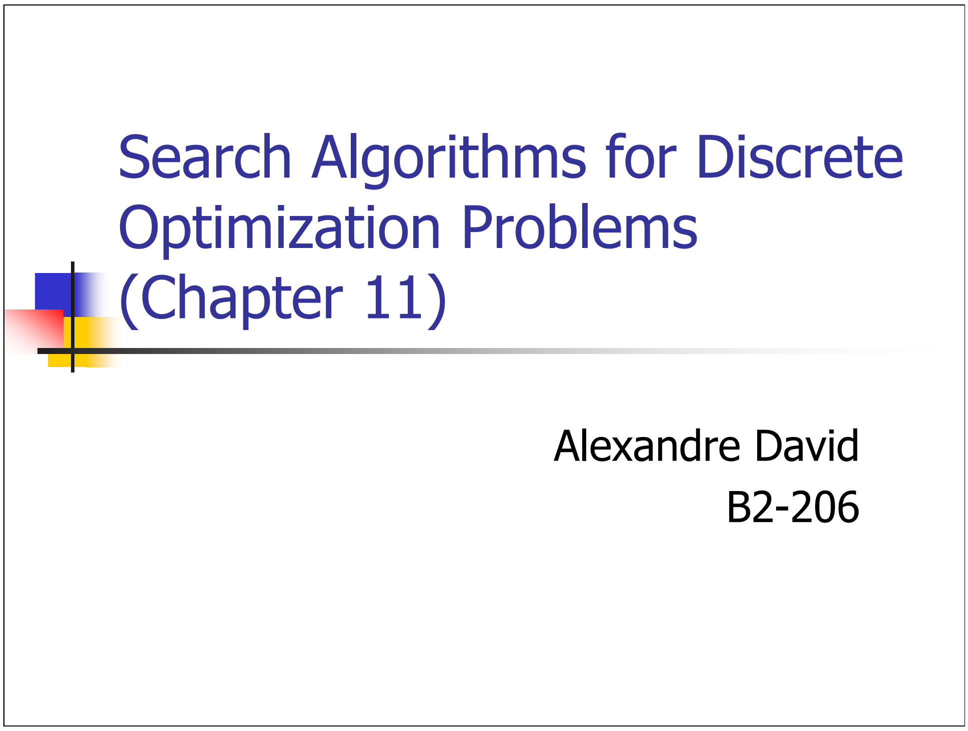


Search Algorithms for Discrete Optimization Problems (Chapter 11)

Alexandre David
B2-206

Discrete optimization problems are also referred as combinatorial problems. They are computationally expensive problems with significant theoretical and practical interests. These algorithms systematically search the space of possible solutions for optimal ones.

Today

- Discrete optimization – basics.
- Sequential search algorithms.
- Parallel depth-first search.
- Parallel best-first search.
- Speedup anomalies.

Discrete Optimization Problems (DOP)

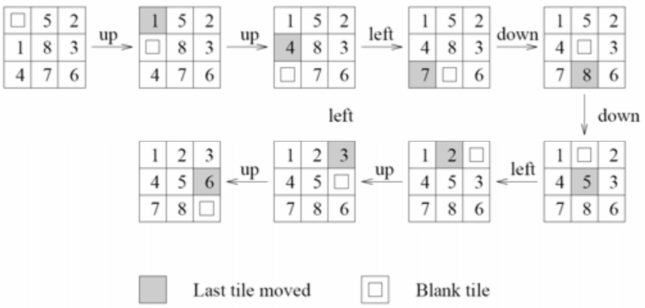
- Tuple (*S*,*f*) where
  - *S* is a finite (or countable) set of feasible solutions.
  - The function *f* is the cost *f*: $S \rightarrow R$.
- Objective: Find a solution $x_{opt} \in S$ s.t. $f(x_{opt}) \leq f(x)$ for all $x \in S$.
- Applications: Planning, scheduling, layout of VLSI chips, etc …

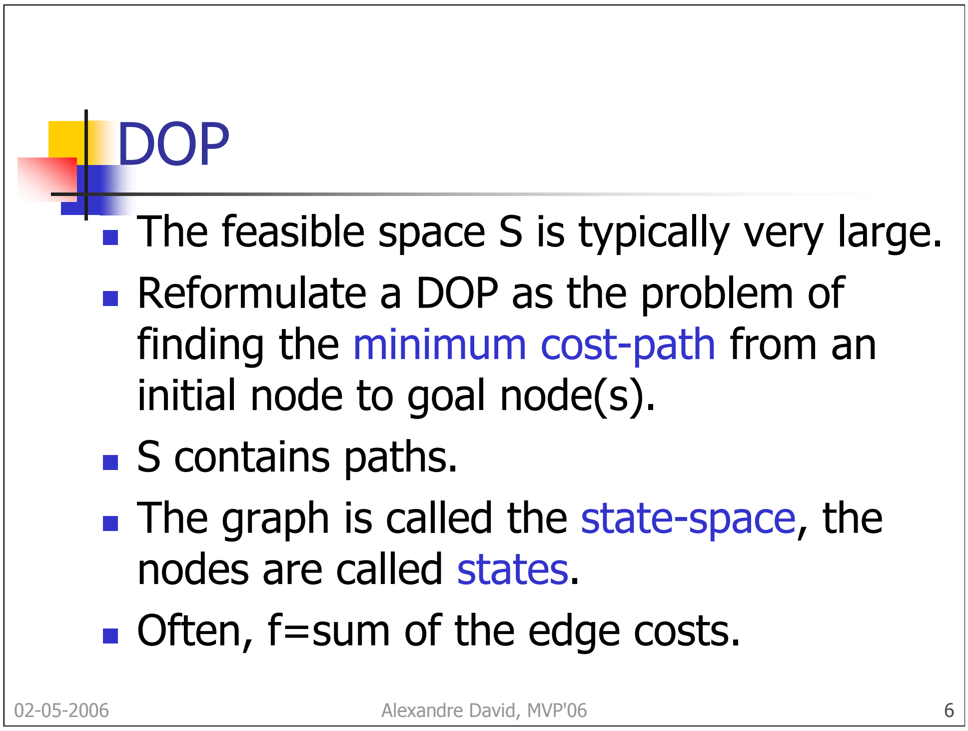
The 0/1 Integer-Linear-Programming Problem

- Input: an $m * m$ matrix A, an $m * 1$ vector b, and an $n * 1$ vector c.
- Find vector $\overline{x}$ of 0/1 s.t.
  - The constraint $A\overline{x} \geq b$ is satisfied.
  - The function $f(\overline{x}) = c^T\overline{x}$ is minimized.

The 8-Puzzle Problem

S = All paths from initial
to final configurations.
Function *f* =number of moves.





DOP

- The feasible space S is typically very large.
- Reformulate a DOP as the problem of finding the minimum cost-path from an initial node to goal node(s).
- S contains paths.
- The graph is called the state-space, the nodes are called states.
- Often, f=sum of the edge costs.

A terminal node has no successor. All the other nodes are non-terminal nodes.

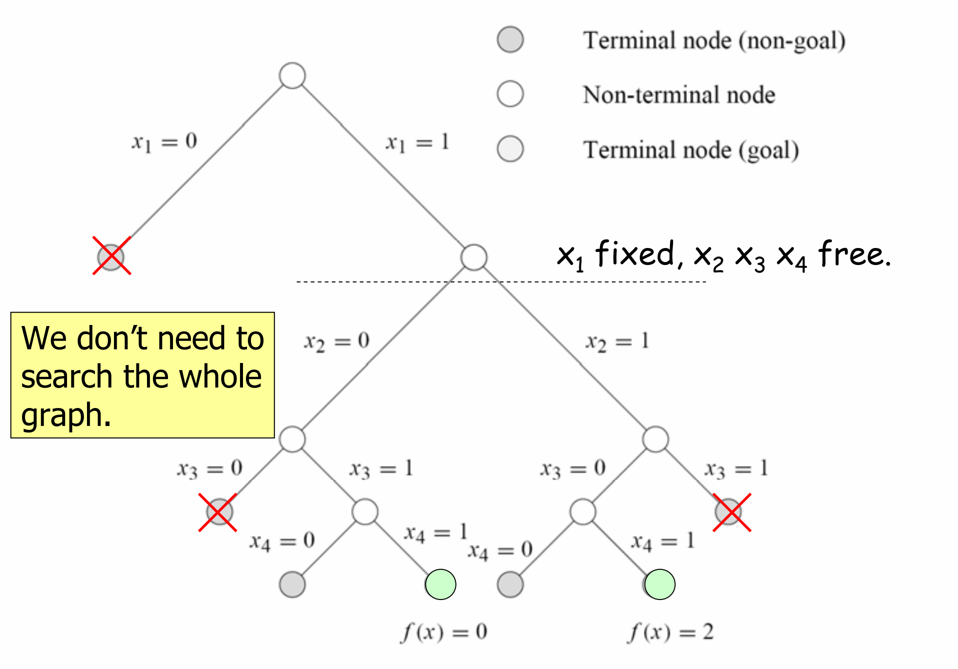
Point of reformulating a DOP as a graph search problem: Can be solved using branch-and-bound & other search algorithm to avoid searching the whole set S.

0/1 Integer-Linear-Programming Problem Revisited

$$A=\begin{bmatrix} 5 & 2 & 1 & 2 \\ 1 & -1 & -1 & 2 \\ 3 & 1 & 1 & 3 \end{bmatrix} \quad b=\begin{bmatrix} 8 \\ 2 \\ 5 \end{bmatrix} \quad c=\begin{bmatrix} 2 \\ 1 \\ -1 \\ -2 \end{bmatrix}$$

$$\longrightarrow \boxed{\begin{array}{l} 5x_1 + 2x_2 + x_3 + 2x_4 \geq 8 \\ x_1 - x_2 - x_3 + 2x_4 \geq 2 \\ 3x_1 + x_2 + x_3 + 3x_4 \geq 5 \end{array}} \quad \boxed{\text{Constraints}}$$

$$\longrightarrow \boxed{f(x) = 2x_1 + x_2 - x_3 - 2x_4} \quad \boxed{\text{Cost}}$$



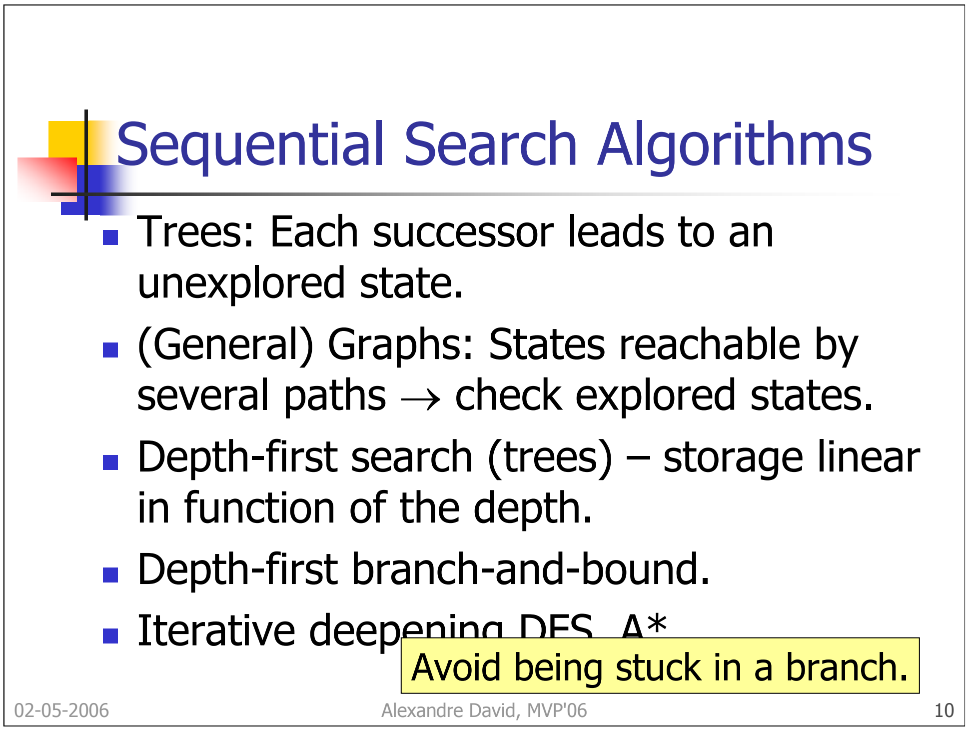
x_1 fixed, x_2 x_3 x_4 free.

We don't need to search the whole graph.

Figure 11.2 The graph corresponding to the 0/1 integer-linear-programming problem.

Heuristics

- Often possible to estimate the cost to reach goal states from an intermediate state.
  - Heuristic estimate.
  - If the heuristic is guaranteed to be a lower bound on the cost then it is an admissible heuristic.
  - Good for pruning the search.
- 8-puzzle problem: Manhattan distance.



Sequential Search Algorithms

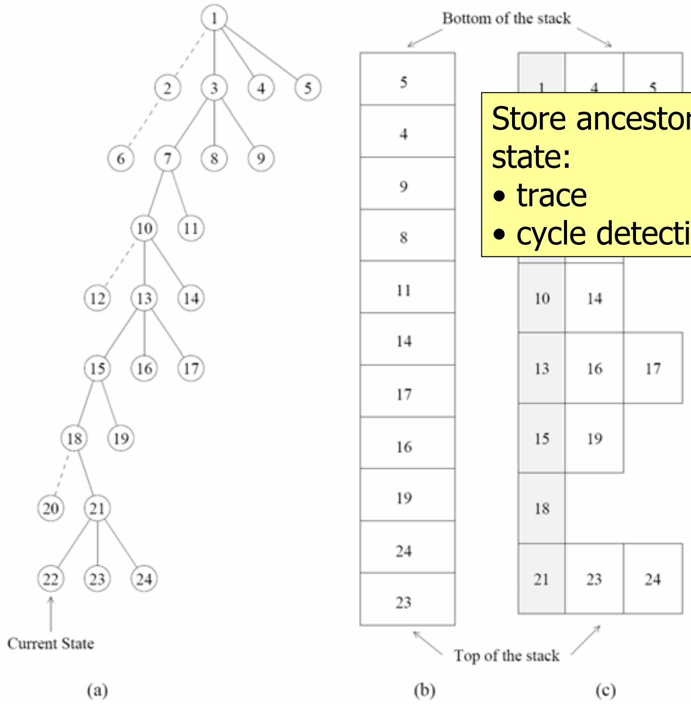
- Trees: Each successor leads to an unexplored state.
- (General) Graphs: States reachable by several paths → check explored states.
- Depth-first search (trees) – storage linear in function of the depth.
- Depth-first branch-and-bound.
- Iterative deepening DFS, A*

Avoid being stuck in a branch.

Iterative deepening: Fix a max depth and increase it if solutions were not found to search again. Finds paths but not least cost paths.

Iterative deepening A*: Same principle but with a cost bound. Finds optimal solutions if the heuristic function is admissible.

DFS



Store ancestor state:
- trace
- cycle detection.

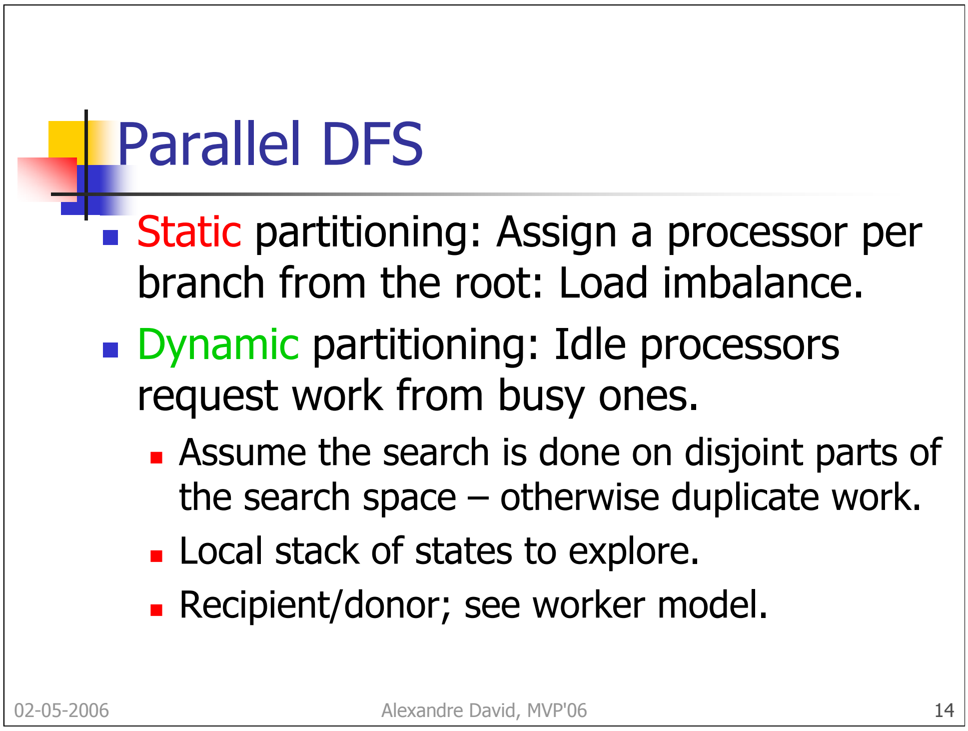
Best First Search

- 2 lists:
  - States to be explored on the open list. waiting
  - States explored on the closed list. passed
  - Choose best from open list, replace if find better states – more memory.
- A* algorithm:
  - $l(x)=g(x)+h(x)$ used to order the search.
  - $g(x)$: from init to x.
  - $h(x)$: from x to goal.

Worse memory complexity: Proportional to the number of states explored, not the depth.

Sequential vs. Parallel Search

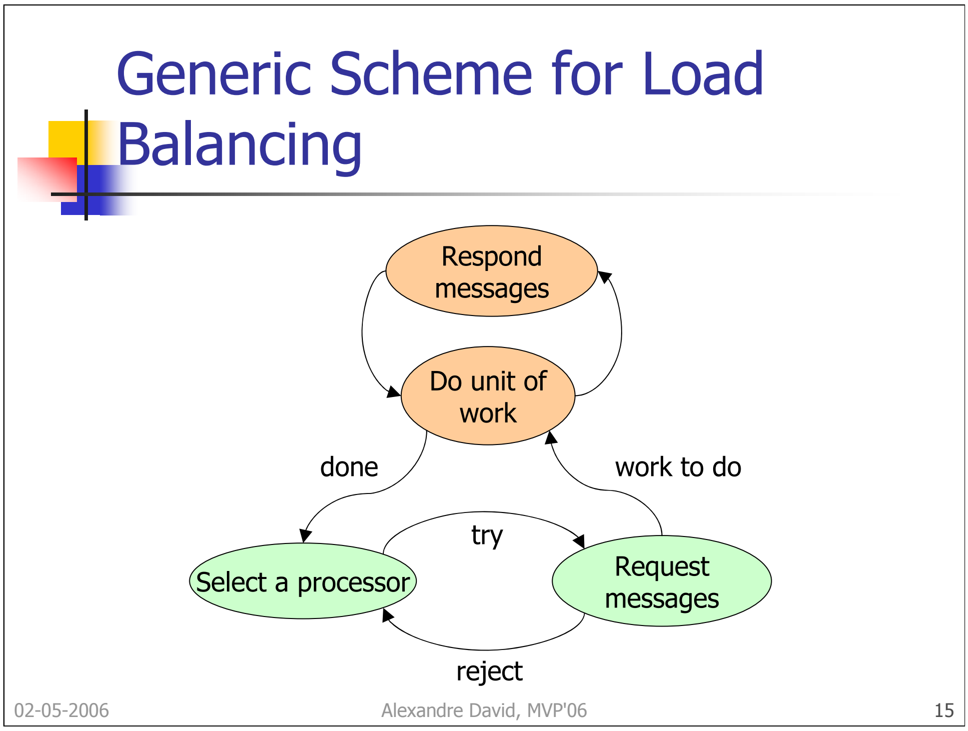
- Overhead for parallel search (as usual communication, contention, load imbalance).
- Big difference with other algorithms: Amount of work can be very different because different parts of the search space are explored.
  - Super-linear anomalies.
  - Critical issue: Distribution of the search space.



Parallel DFS

- Static partitioning: Assign a processor per branch from the root: Load imbalance.
- Dynamic partitioning: Idle processors request work from busy ones.
  - Assume the search is done on disjoint parts of the search space – otherwise duplicate work.
  - Local stack of states to explore.
  - Recipient/donor; see worker model.

When a processor finds the goal, all processors are stopped.



Generic Scheme for Load Balancing

Respond messages

Do unit of work

done

work to do

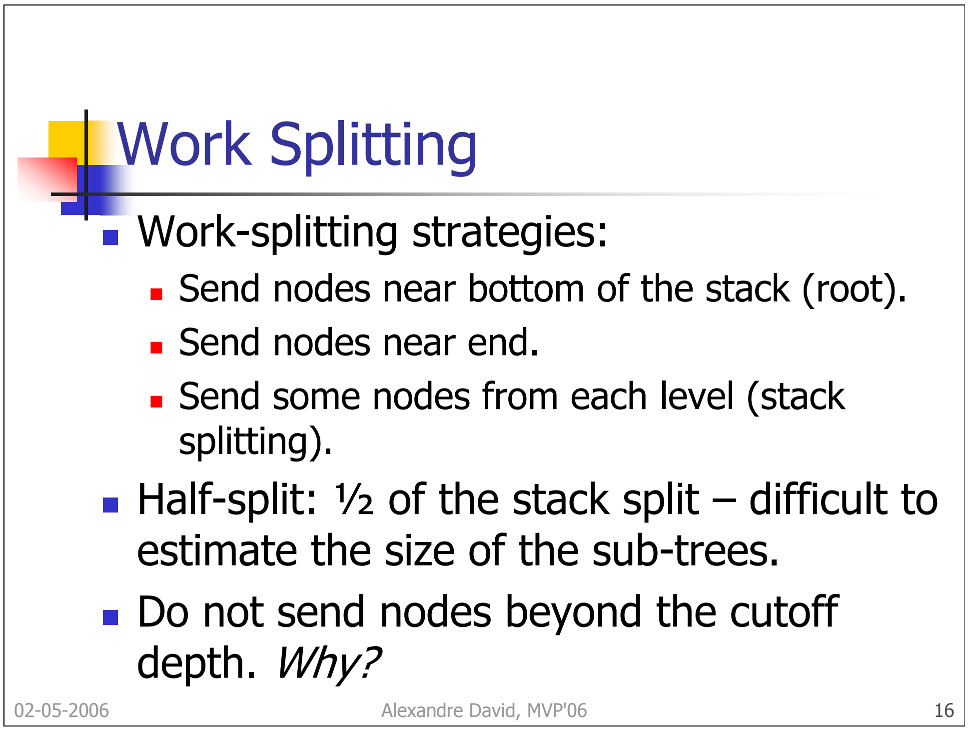
try

Select a processor

Request messages

reject

2 modes: Processor active (orange) or inactive (green), w.r.t. computations.



Work Splitting

- Work-splitting strategies:
  - Send nodes near bottom of the stack (root).
  - Send nodes near end.
  - Send some nodes from each level (stack splitting).
- Half-split: ½ of the stack split – difficult to estimate the size of the sub-trees.
- Do not send nodes beyond the cutoff depth. *Why?*

We don’t want to have either the donor or the recipient to become idle too soon.

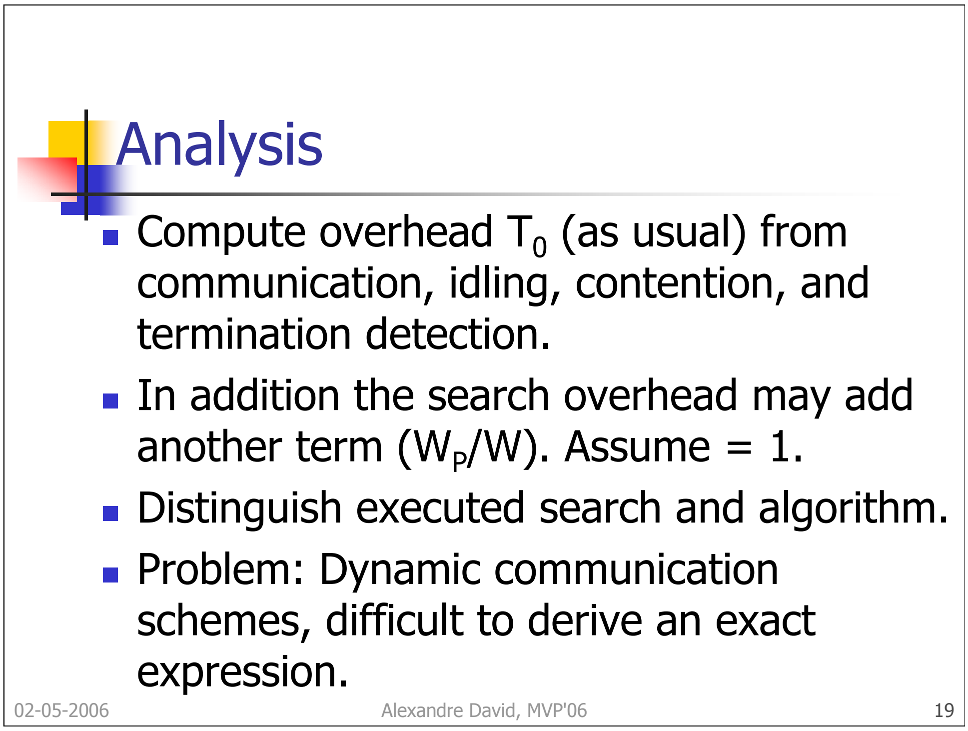
Usually, sub-trees are larger near the root than near the cutoff depth.

Load Balancing

- Which processor to ask?
  - Asynchronous Round Robin.
    - Ask to (local_target++)%p.
    - + asynchronous, - even work.
  - Global Round Robin.
    - Ask to (global_target++)%p.
    - - contention, + even work.
  - Random Polling.
    - + + ?

Analysis

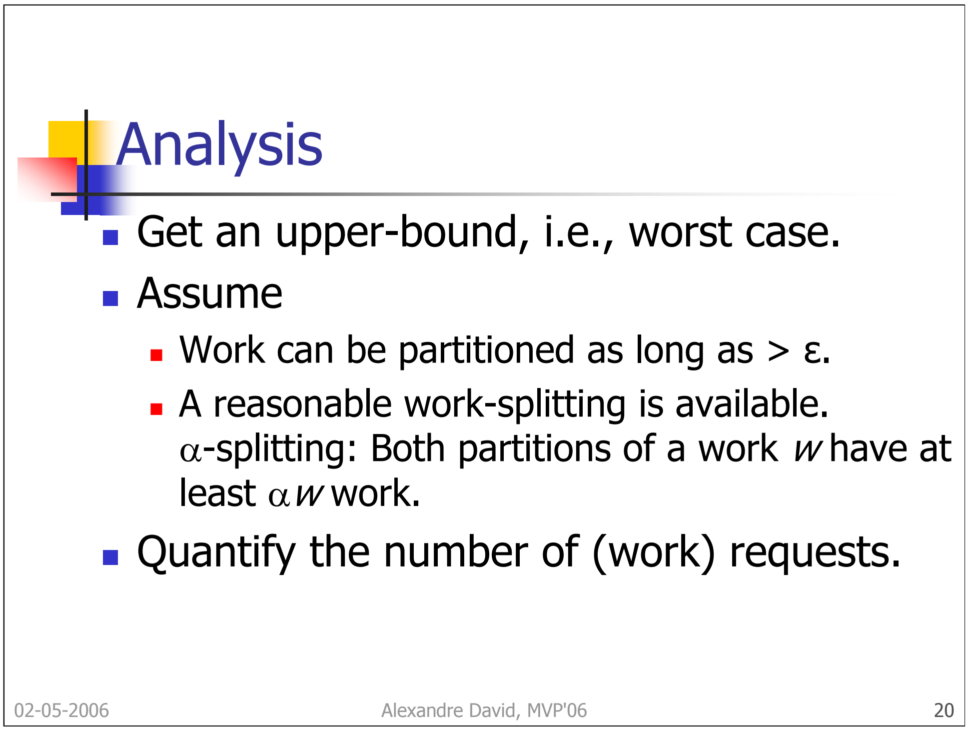
- How to analyze?
- What's W? W_P?
- Problem:
  - The execution time depends on the search primarily (and secondarily on the size of the input).



Analysis

- Compute overhead T_0 (as usual) from communication, idling, contention, and termination detection.
- In addition the search overhead may add another term (W_P/W). Assume = 1.
- Distinguish executed search and algorithm.
- Problem: Dynamic communication schemes, difficult to derive an exact expression.

Idling time negligible to communication time.

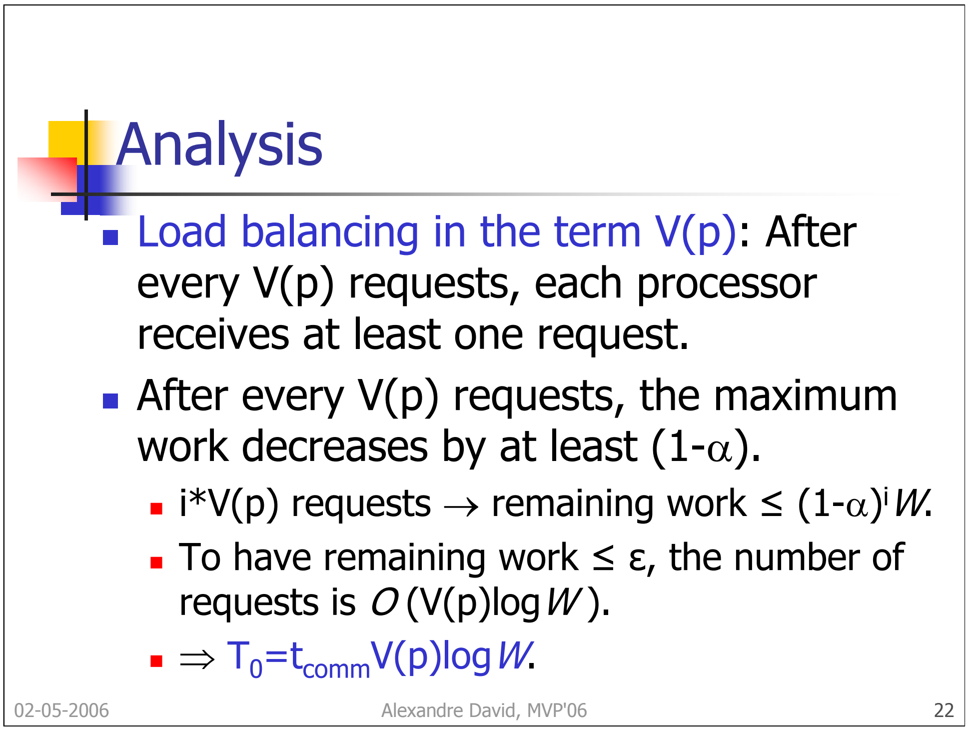


Analysis

- Get an upper-bound, i.e., worst case.
- Assume
  - Work can be partitioned as long as > ε.
  - A reasonable work-splitting is available. α-splitting: Both partitions of a work w have at least αw work.
- Quantify the number of (work) requests.

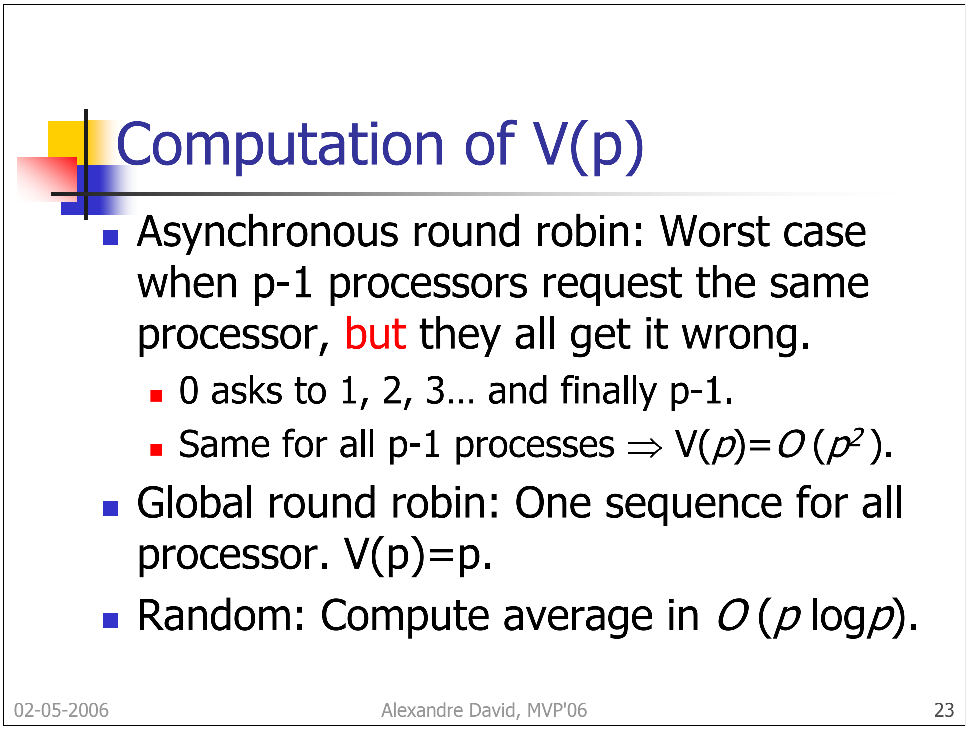
Analysis

- Donor has $w_i \to w_j + w_k$.
- Assumption: $w_j > \alpha w_i$, $w_k > \alpha w_i$.
- After transfer, donor and recipient have $\leq (1-\alpha)w_i$.
- $w_0,\ldots,w_{p-1} \leq w$. Split all (2p pieces), largest $\leq (1-\alpha)w$.
- If every processor gets a request once, then each piece has been split once $\Rightarrow$ maximum load reduced by $(1-\alpha)$ at any processor.



Analysis

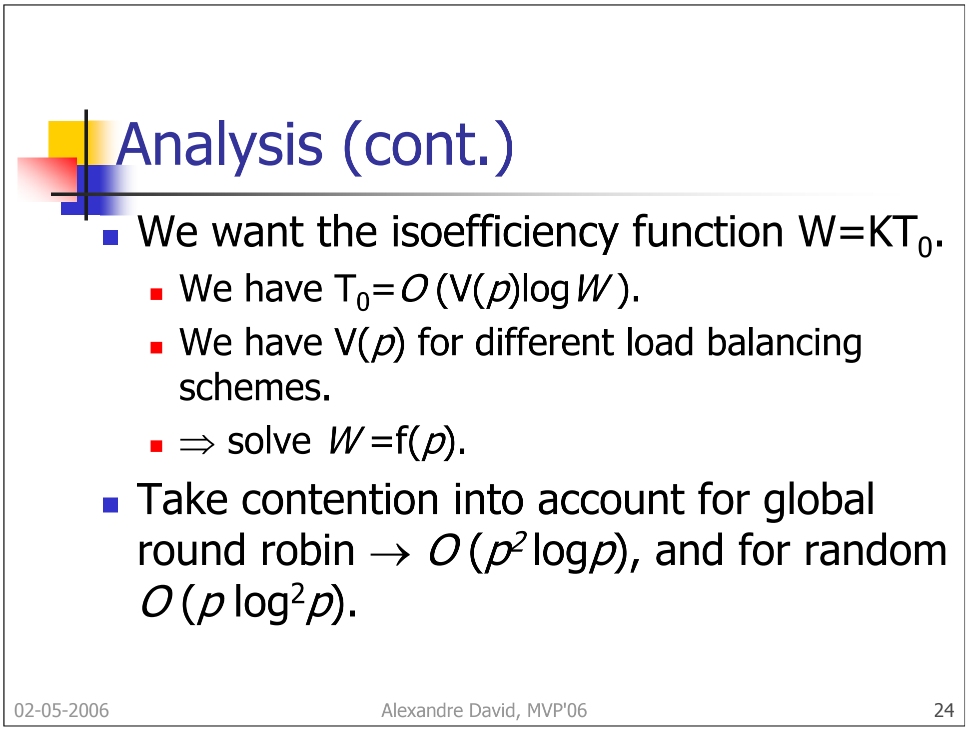
- Load balancing in the term V(p): After every V(p) requests, each processor receives at least one request.
- After every V(p) requests, the maximum work decreases by at least $(1-\alpha)$.
  - i*V(p) requests → remaining work $\leq (1-\alpha)^i W$.
  - To have remaining work $\leq \varepsilon$, the number of requests is $O(V(p)\log W)$.
  - $\Rightarrow T_0 = t_{comm} V(p) \log W$.



Computation of V(p)

- Asynchronous round robin: Worst case when p-1 processors request the same processor, but they all get it wrong.
  - 0 asks to 1, 2, 3… and finally p-1.
  - Same for all p-1 processes ⇒ $V(p)=O(p^2)$.
- Global round robin: One sequence for all processor. V(p)=p.
- Random: Compute average in $O(p \log p)$.

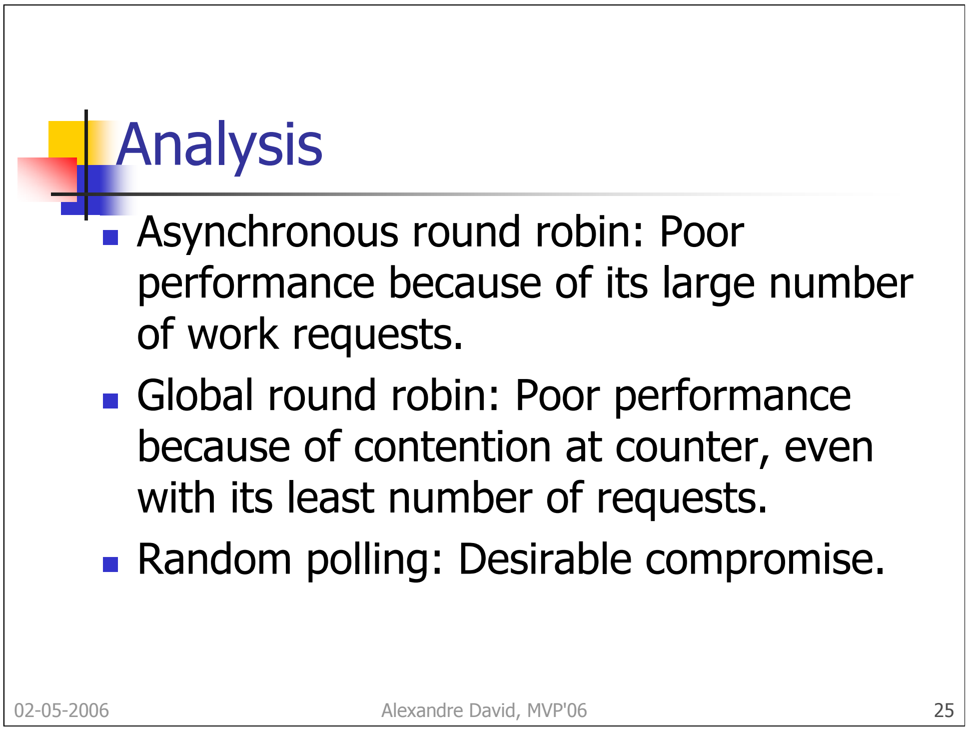
Personally, I don't agree with the upper bound in the book, I'll rather write $(p-1)^2$.



Analysis (cont.)

- We want the isoefficiency function $W=KT_0$.
  - We have $T_0=O(V(p)\log W)$.
  - We have $V(p)$ for different load balancing schemes.
  - $\Rightarrow$ solve $W=f(p)$.
- Take contention into account for global round robin $\rightarrow O(p^2\log p)$, and for random $O(p\log^2 p)$.

Contention: The global counter must be incremented $O(p \log W)$ times in $O(W/p)$ time.



Analysis

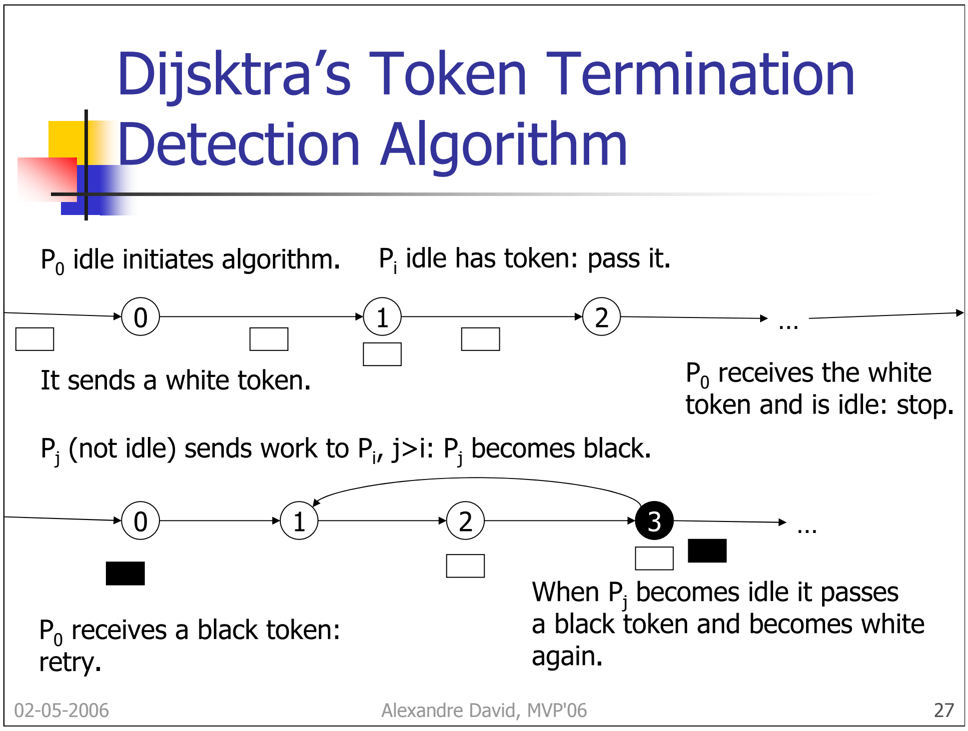
- Asynchronous round robin: Poor performance because of its large number of work requests.
- Global round robin: Poor performance because of contention at counter, even with its least number of requests.
- Random polling: Desirable compromise.

02-05-2006 Alexandre David, MVP'06 25

Random is good sometimes, but it’s a uniform random distribution.

Termination Detection

- Normally simple token based algorithm works but not here. When a processor goes idle, it may receive more work later.
- Dijkstra’s token algorithm.
- Tree-based algorithm.



Dijsktra’s Token Termination Detection Algorithm

P_0 idle initiates algorithm.

P_i idle has token: pass it.

It sends a white token.

P_0 receives the white token and is idle: stop.

P_j (not idle) sends work to P_i, j>i: P_j becomes black.

When P_j becomes idle it passes a black token and becomes white again.

P_0 receives a black token: retry.

Black/white or red/green, whatever.

Tree-Based Termination Detection

- Weight 1 from the root at the start.
- Weights are divided and go down the tree with the work.
- When work is done, weights are returned from the source.
- Terminate when weight is one at the root.
- Careful with precision.

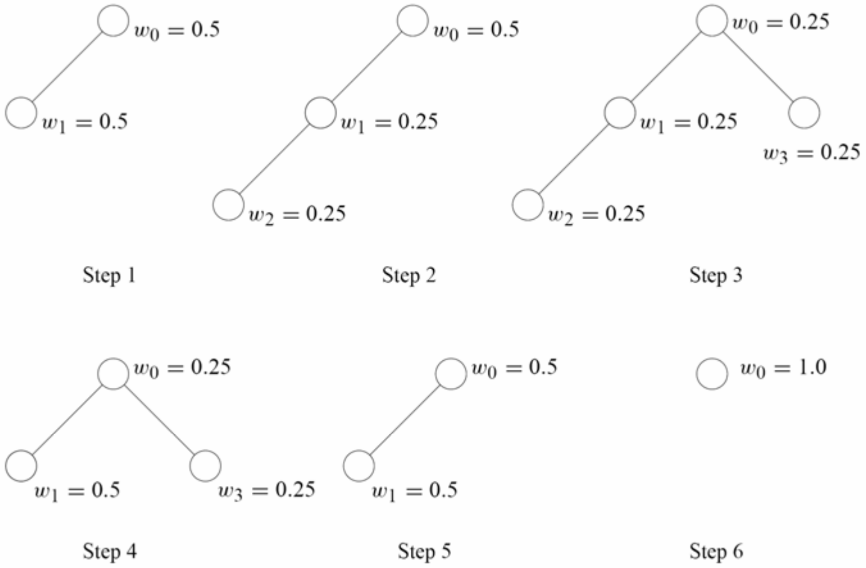


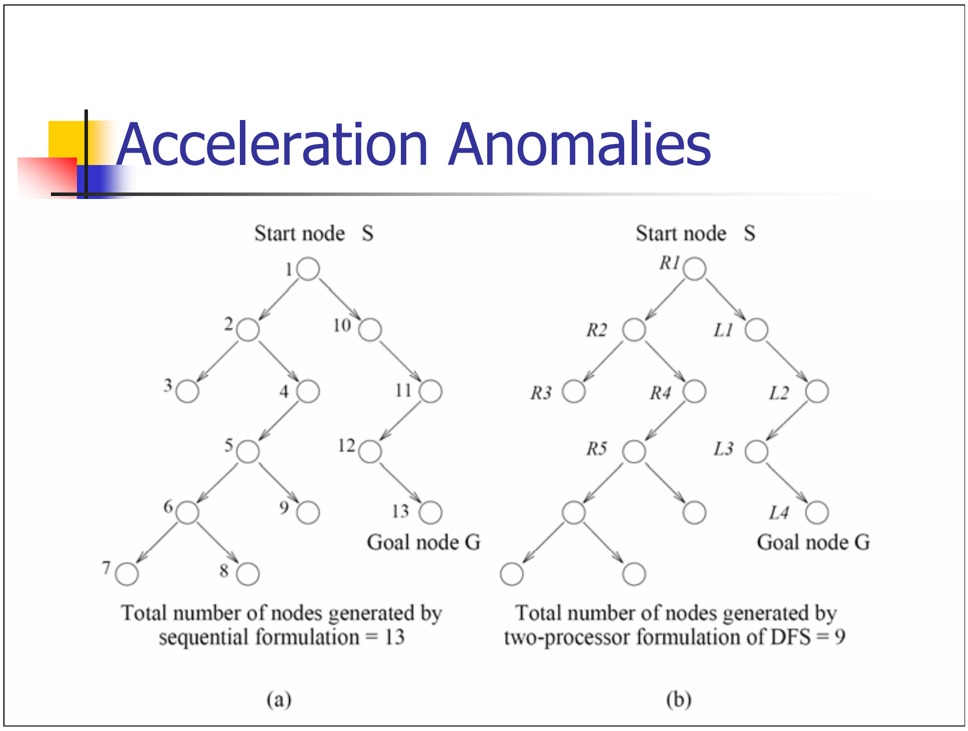
Figure 11.10 Tree-based termination detection. Steps 1–6 illustrate the weights at various processors after each work transfer.

Experiments

Analysis validated by experimental results.
It works. ☺

Parallel Best-First Search

- Avoid bottleneck with one global open list.
- Local open lists must synchronize and share their best nodes.
  - Different communication schemes.
- Distributed cycle detection: Hash nodes to map them on specific processors (local check) but degrades performance.



Acceleration Anomalies

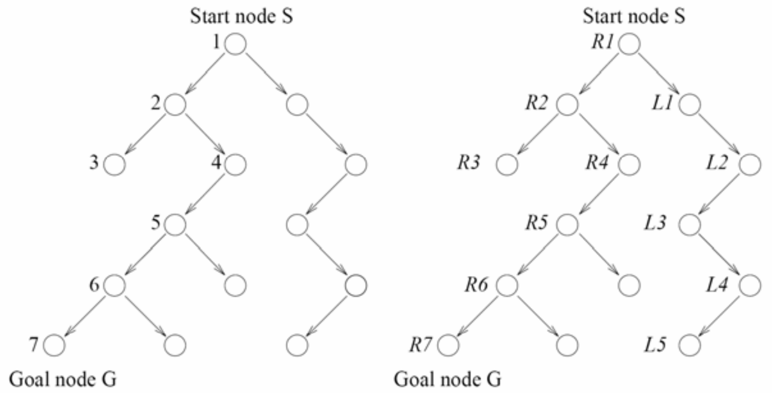
Total number of nodes generated by sequential formulation = 13

(a)

Total number of nodes generated by two-processor formulation of DFS = 9

(b)

Deceleration Anomalies



Total number of nodes generated by sequential DFS = 7

(a)

Total number of nodes generated by two-processor formulation of DFS = 12

(b)