Sorting in Linear Time – Solution

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Proof Complement (8.1-2)

Let's prove the upper-bound. This is trivial:

$$\lg(n!) = \sum_{k=1}^{n} \lg(k) \tag{1}$$

$$\leq \sum_{k=1}^{n} \lg(n) \tag{2}$$

$$= n\lg(n) \tag{3}$$

We conclude $\lg(n!) = O(n\lg(n))$.

Let's prove the lower-bound by using techniques of appendix A.2, which consists in splitting the sum:

$$\lg(n!) = \sum_{k=1}^{n/2} \lg(k) + \sum_{k=n/2+1}^{n} \lg(k) \tag{4}$$

$$\geq \sum_{k=n/2+1}^{n} \lg(n/2+1) \tag{5}$$

$$\geq \sum_{k=n/2+1}^{n} \lg(n/2) \tag{6}$$

$$= \frac{n}{2}\lg(n/2) \tag{7}$$

$$= \frac{n}{2}(\lg(n) - 1) \tag{8}$$

Now we need a trick to get rid of $n/2$. We could try to keep the first sum but that won't help us much. Let's keep "some" of the log only.

$$\lg(n!) \geq \frac{n}{2}(\lg(n) - 1) \tag{9}$$

$$= \frac{n}{4}\lg(n) + \frac{n}{2}(\frac{\lg(n)}{2} - 1) \tag{10}$$

$$\geq \frac{n}{4}\lg(n) \text{ for } n \geq 4 \tag{11}$$

We conclude $\lg(n!) = \Omega(n\lg(n))$, which finally gives us $\lg(n!) = \Theta(n\lg(n))$.