Recurrences – Suggested Solutions

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4.1-1 By induction on n, assuming $T(k) = O(\lg(k))$ for $k \le \lceil n/2 \rceil$:

$$\begin{aligned} T(n) &\le c\lg(\lceil n/2 \rceil) + 1 \\ &\le c\lg(n/2+1) + 1 \\ &\le c\lg(n+2) - c + 1 \end{aligned}$$

We want this expression to be $\le c\lg n$. Let's see if it is possible and for which c:

$$\begin{aligned} & c\lg(n+2) - c + 1 \le c\lg n \\ \Leftrightarrow\quad & 1 \le c(\lg n - \lg(n+2) + 1) \\ \Leftrightarrow\quad & 1 \le c(\lg\tfrac{n}{n+2} + 1) \end{aligned}$$

We need a c such that this holds for all $n \ge n_0$ from which we can apply the induction. We note that $\lg\frac{n}{n+2}$ is monotonic increasing, so if the expression holds for n_0 (yet to find), then it will hold for $n \ge n_0$. For $n = 2$ it does not work: $1 + \lg(2/4) = 0$; for $n = 3$: $1 + \lg(3/5) = 0.26\ldots$, we can find a c. For $c \ge \frac{1}{1+\lg(3/5)}$ we have:

$$T(n) \le c\lg n$$

which complete the proof of the induction step. The base case is problematic as on page 64: check for $n = 2$ and $n = 3$ assuming $T(1) = 1$ and choose c large enough. Conclude by induction that $T(n) = O(\lg(n))$ for all $n \ge 2$.

4.1-2 By induction on n assuming $T(k) = \Omega(k\lg k)$ holds for $k \le \lfloor n/2 \rfloor$:

$$\begin{aligned} T(n) &\ge 2c(\lfloor n/2 \rfloor)\lg(\lfloor n/2 \rfloor) + n \\ &\ge 2c(n/2-1)\lg(n/2-1) + n \\ &\ge c(n-2)\lg(n-2) - c(n-2) + n \\ &\ge c(n-2)\lg(n/2) - c(n-2) + n \text{ (for } n \ge 4) \\ &\ge c(n-2)\lg n - 2c(n-2) + n \\ &\ge cn\lg n - 2c\lg n - 2c(n-2) + n \\ &\ge cn\lg n + n(1-2c) + 4c - 2c\lg n \end{aligned}$$

We choose $c = 1/3$ and we examine $f(x) = 3 * (x(1-2c) + 4c - 2c\lg x) = x + 4 - 2\lg x$. $f'(x) = 1 - 2/x \ge 0 \Leftrightarrow x \ge 2$.

$$\begin{cases} f(2) = 2 + 4 - 2 > 0 \\ f'(x) \ge 0 \text{ for } x \ge 2 \\ f \text{ is continuous} \end{cases} \Rightarrow f(x) \ge 0 \text{ for } x \ge 2 \qquad (1)$$

Thus for $n \geq 2$ and $c = 1/3$ we have $n(1-2c) + 4c - 2c\text{lg}n \geq 0$ and therefore $T(n) \geq cn\text{lg}n$. Base case $n = 1$: $T(1) \geq 0$. Conclude by induction that $T(n) = \Omega(n\text{lg}n)$ for $n \geq 1$. Use theorem 3.1 to conclude $T(n) = \Theta(n\text{lg}n)$.

4.2-1 The recursion tree for $T(n) = 3T(\lfloor n/2 \rfloor) + n$ is given in Fig. 1. The total sum is $T(n) = n\sum_{i=0}^{\text{lg}(n)-1}(3/2)^i + n^{\text{lg}(3)} = O(n^{\text{lg}(3)})$.

We take $n^{\text{lg}(3)}$ as the guess for the upper-bound but to verify this bound, lets make the following induction hypothesis: $\exists c > 0, \exists b > 0, \exists n_0, \forall k \geq n_0 \wedge k < n \, : \, T(k) \leq ck^{\text{lg}(3)} - bk$.

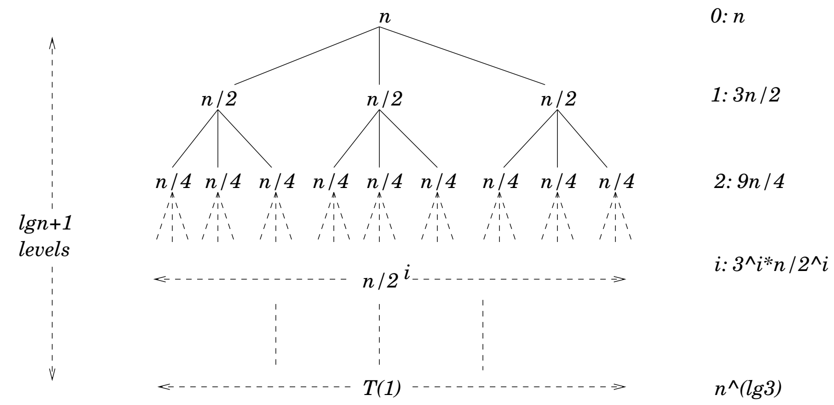

Figure 1: Recursion tree of 4.2-1.

Let's prove the induction step:

$$\begin{aligned}
T(n) &= 3T(\lfloor n/2 \rfloor) + n \\
&\leq 3c(\lfloor n/2 \rfloor)^{\text{lg}(3)} - 3b(\lfloor n/2 \rfloor) + n \quad (\textit{substitution}) \\
&\leq 3c(n/2)^{\text{lg}(3)} - 3b(n/2 - 1) + n \\
&= 3c * 3^{\text{lg}(n)-1} - 3b(n/2 - 1) + n \\
&= c * 3^{\text{lg}(n)} - 3b(n/2 - 1) + n \\
&= cn^{\text{lg}(3)} - bn - bn/2 + n + 3b
\end{aligned}$$

Let's choose $b = 30$: $n(1 - b/2) + 3b = 90 - 14n \leq 0 \Leftrightarrow n \geq 45/7$. For $n_0 = 7$ and $b = 30$ we have $T(n) \leq cn^{\text{lg}(3)} - bn$, which proves the induction step. Now we need to prove $T(7) \leq c * 7^{\text{lg}(3)} - 7 * 30$ for some $c > 0$, which is left as an exercise for the reader. It is clear that by choosing c large enough, this will hold, which completes the induction proof. We have proven $\exists c > 0, \exists b > 0, \exists n_0, \forall n \geq n_0 \, : \, T(n) \leq cn^{\text{lg}(3)} - bn$, which then proves that $T(n) = O(n^{\text{lg}(3)})$.