Sorting in Linear Time

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B2-206

Linear Sort? But…

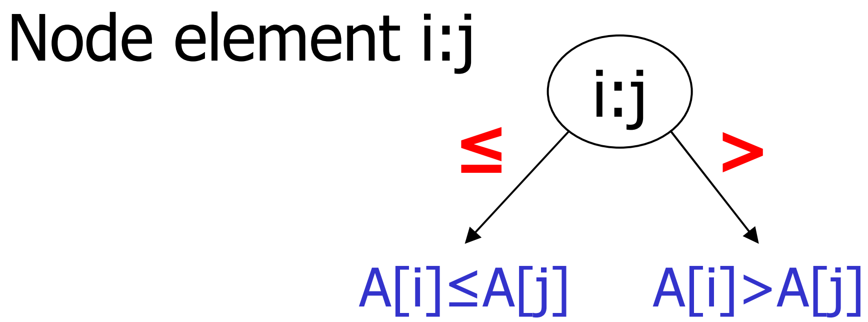
- Best algorithms so far perform in $\Theta(n \lg n)$. But they are comparison sorts.
  - We do not inspect the value or use other information.
  - Only comparisons between keys.
- Comparison sorts need at least $\Omega(n \lg n)$. Previous algorithms were optimal!

Optimality

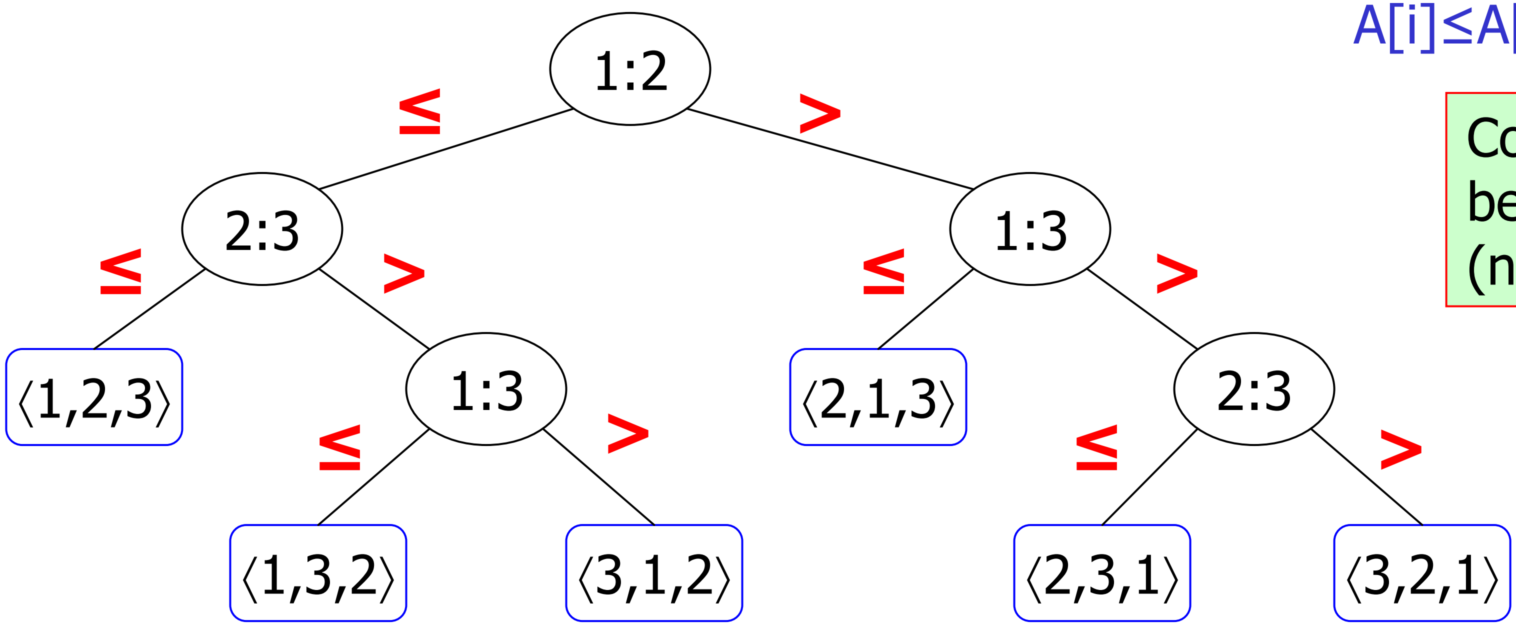
- How to prove that $n \lg n$ is the lower bound for *all possible* comparison sort algorithms?
- Use decision-tree.
  - Binary trees representing comparisons.
  - All possible permutations represented.
  - $n!$ permutations, thus $n!$ leaves.
  - Sorting algorithms find an ordering, i.e., a path.

Decision Tree

"The tree represents the comparisons done by a sorting algorithm."



Comparisons between all (needed) pairs.

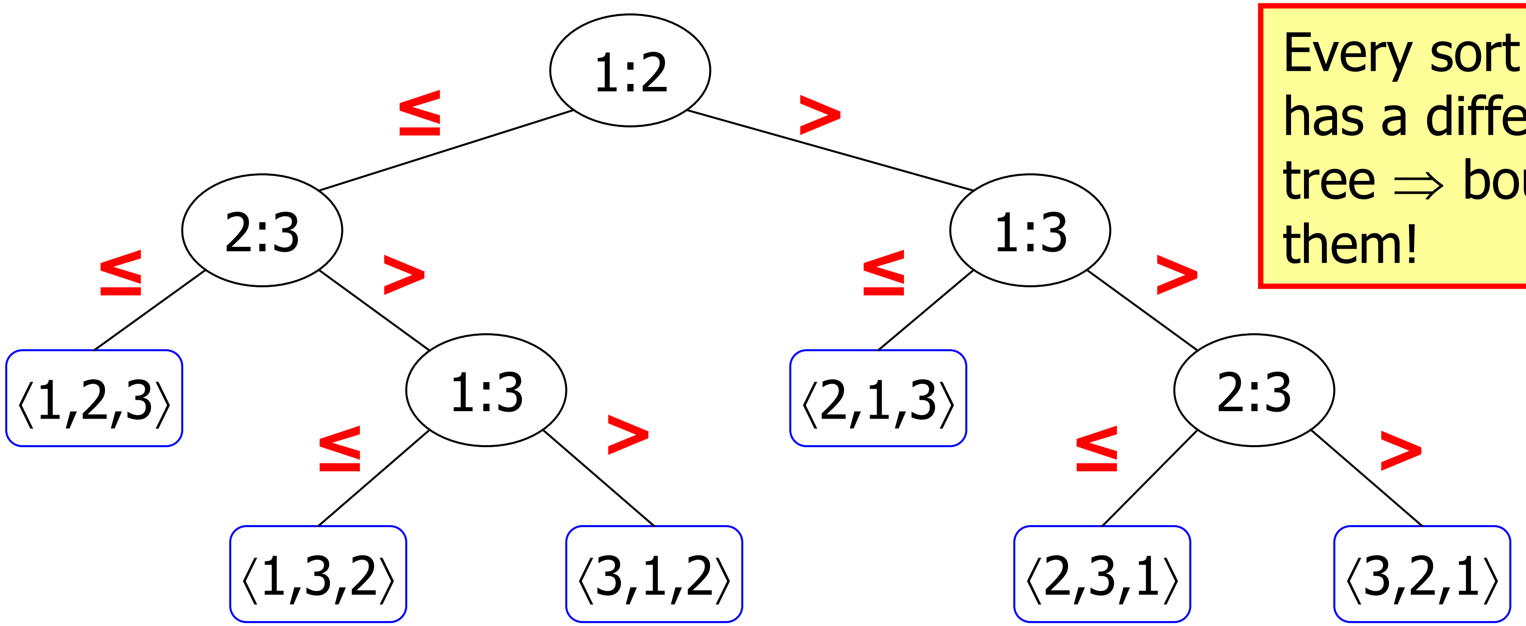


Decision Tree

The point: Lower bound on the heights of all decision trees = lower bound of running time of any comparison sort algorithm.

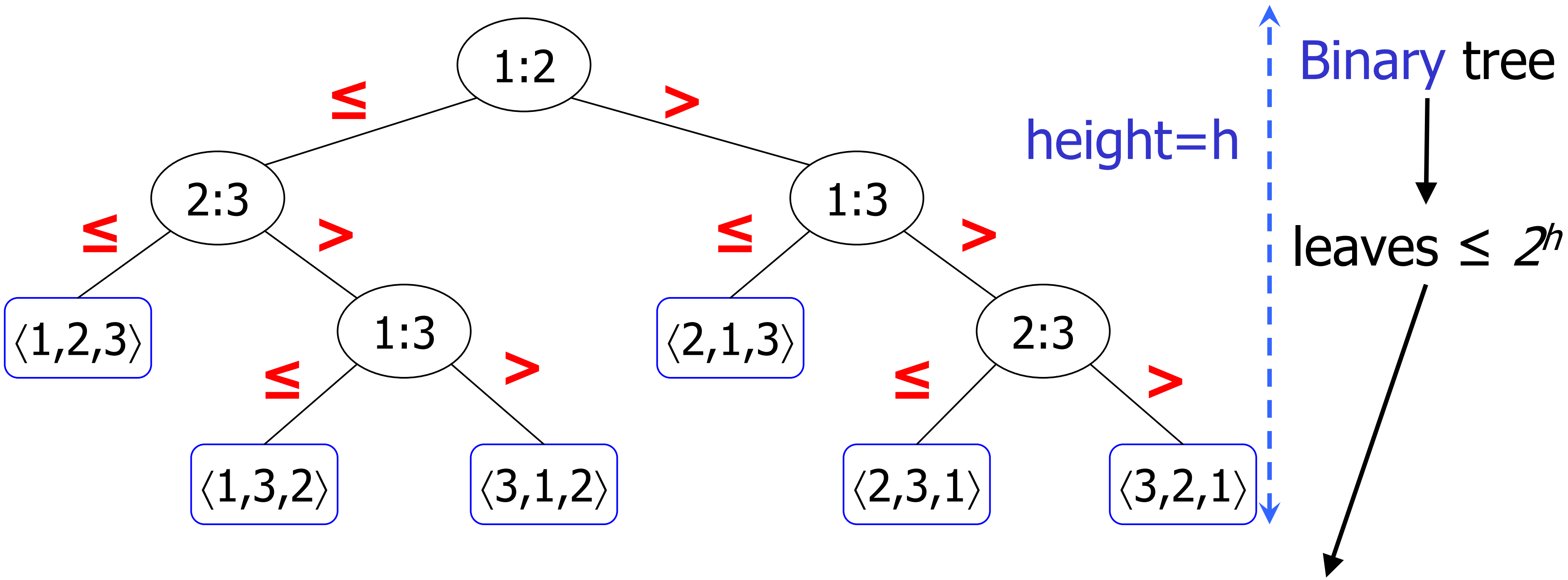
Because to find one ordering one must go on a path from the root to a leaf.

Every sort algorithm has a different decision tree ⇒ bound all of them!



Bound on the height.

Decision Tree



Leaves=permutations ⟶ $n!$ = leaves ⟶ $h \geq \lg(n!) = \Omega(n \lg n)$

(Any correct algorithm must be able to produce any permutation)

(3.18)

Optimality

- Conclusion:
  - Any correct algorithm must go through $\Omega(n \lg n)$ to produce any ordering.
  - We have sorting algorithms that have a bound of $O(n \lg n)$.
  - These (comparison) sorting algorithm are optimal!
  - You can't do better. If you do better, then your algorithm cannot generate all the permutations and is incorrect.

Counting Sort

- Assume that the input is made of integers with a small range (k):
  - $0 \le a_{1..n} \le k$.
  - When $k = O(n)$, counting-sort runs in $\Theta(n)$.
- Idea: For every x
  - count how many elements are ≤x, say *t*,
  - put x at position *t*, the *right* position.

Counting Sort

```
Count_sort(A,B,k):                          // B = output
for i = 0 to k do C[i] = 0                  // initialize
for i = 1 to length(A) do C[A[i]]++         // count x_i
for i = 1 to k do C[i] += C[i-1]            // count ≤x_i
for i = length(A) downto 1 do
    B[C[A[i]]] = A[i]                       // write x at t
    C[A[i]]--                               // update counter
done
```

Needs extra memory with k elements: C[0..k] for counting.

Counting Sort

```
Count_sort(A,B,k):
1 for i = 0 to k do C[i] = 0
2 for i = 1 to length(A) do C[A[i]]++
3 for i = 1 to k do C[i] += C[i-1]
4 for i = length(A) downto 1 do
      B[C[A[i]]] = A[i]
      C[A[i]]--
done
```

	1	2	3	4	5	6	7	8
A:	2	5	3	0	2	3	0	3

	0	1	2	3	4	5	
C:	0	0	0	0	0	0	1
C:	2	0	2	3	0	1	2
C:	2	2	4	7	7	8	3
C:	2	2	4	7	7	8	4
C:	2	2	4	6	7	8	
C:	1	2	4	6	7	8	
C:	1	2	4	5	7	8	

A[8] 3 at C[3] 7 — one less 3: C[3]--

A[7] 0 at C[0] 2 — one less 0: C[0]--

A[6] 3 at C[3] 6 — one less 3: C[3]--

	1	2	3	4	5	6	7	8
B:		0				3	3	

Counting Sort

```
Count_sort(A,B,k):
for i = 0 to k do C[i] = 0
for i = 1 to n do C[A[i]]++
for i = 1 to k do C[i] += C[i-1]
for i = n downto 1 do
    B[C[A[i]]] = A[i]
    C[A[i]]--
done
```

- Running time $T(n)=\Theta(n+k)$.
  For $k=O(n)$, $T(n)=\Theta(n)$.
- There is no comparison.
- The sort is stable (order kept for $a_i==a_j$).
- The sort is not in-place.
- Problem: Range of numbers translates into the size of the working array (counters).

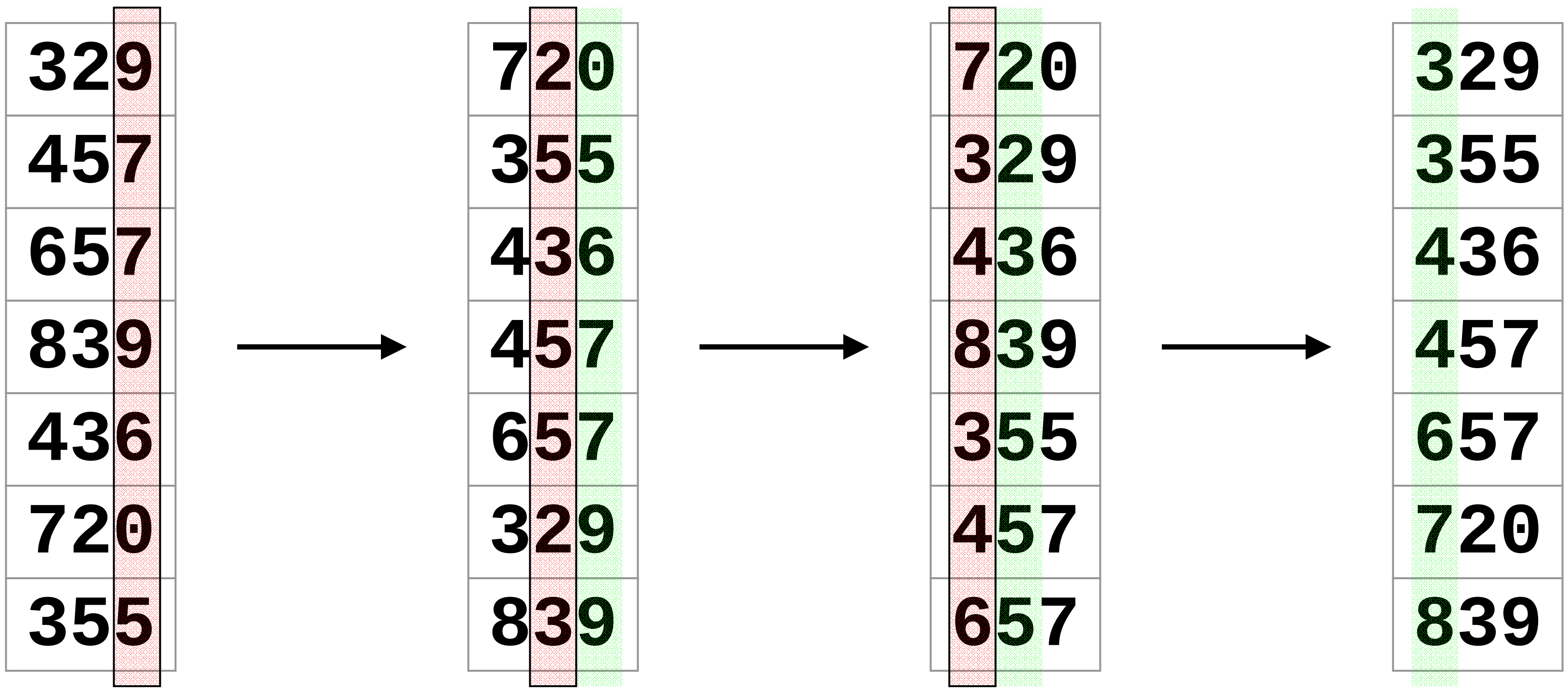
Radix Sort

- Sort on the digits of the numbers.
  - Least significant digits first.
  - Use a **stable** sort (like counting sort).

```
Radix_sort(A,d):
for i = 1 to d do
    stable_sort A with keys=digit i
done
```

- Sort n-digits numbers with each digit taking k values (base k), running time is $T(n)=\Theta(d(n+k))$.

Radix Sort


329
457
657
839
436
720
355
720
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436
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657
329
839
720
329
436
839
355
457
657
329
355
436
457
657
720
839

Bucket Sort

- Assume the input is uniformly distributed.
- Assume values in [0,1).

```
Bucket_sort(A):
n = length(A)
for i = 1 to n do insert A[i] into list B[n*A[i]]
for i = 0 to n-1 do insertion_sort list B[i]
concatenate lists B[0],B[1],…B[n-1]
```

- Running time: $T(n) = \Theta(n) + \sum_{i=0}^{n-1} O(n_i^2)$
- Expected running time: *Θ(n)*.

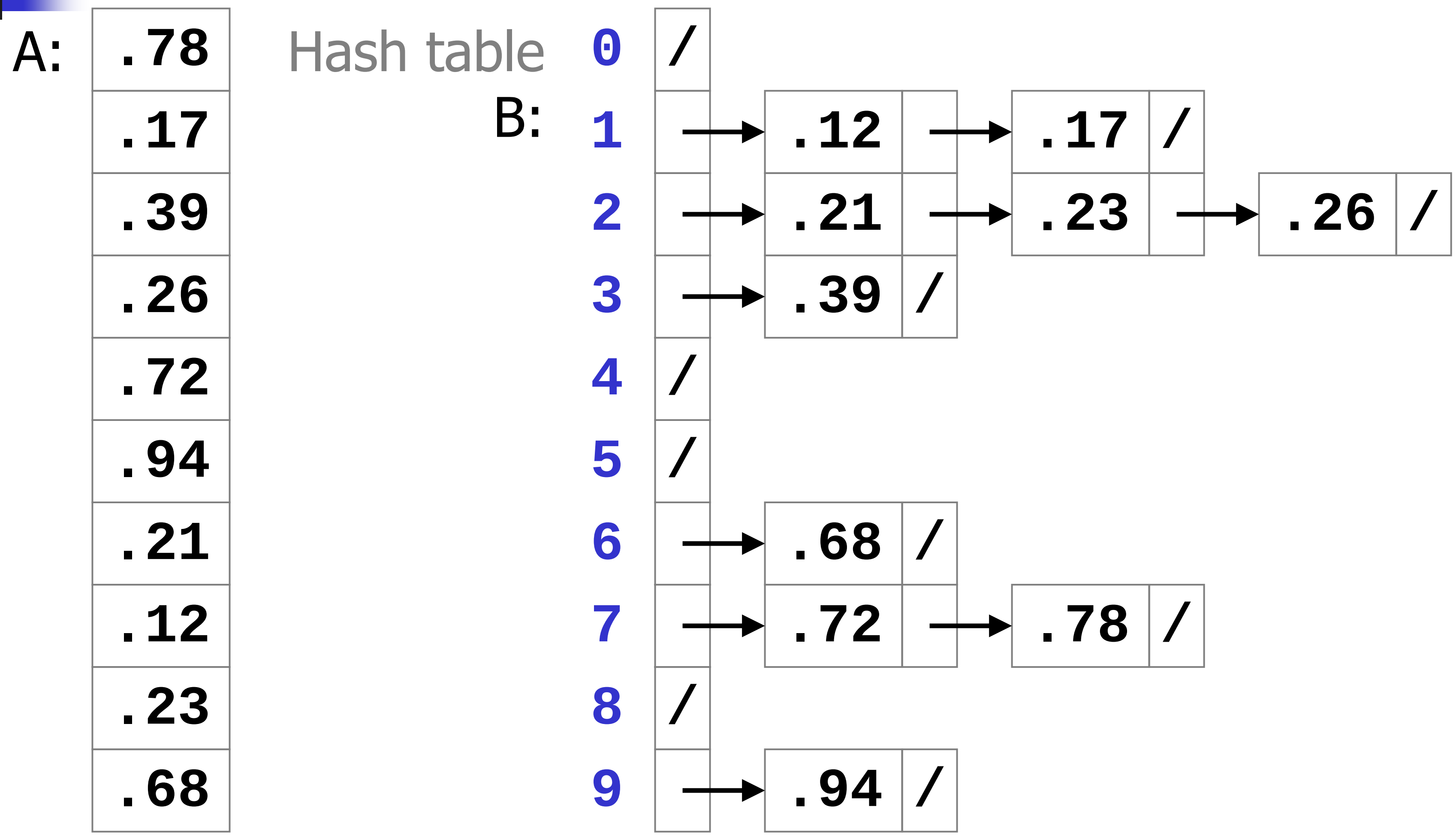
Bucket Sort

- Expected time:
  - Use $E[n_i^2]=2-1/n$.
    See book for technicality.

$$E[T(n)] = \Theta(n) + \sum_{i=0}^{n-1} O(E[n_i^2]) = \Theta(n)$$

  - Idea: *n* elements distributed uniformly in *n* entries $\Rightarrow$ 1 element per entry in average.
    But not *O(1)* for sorting…

Bucket Sort


A:
.78
.17
.39
.26
.72
.94
.21
.12
.23
.68
Hash table
B:
0 /
1 .12 .17 /
2 .21 .23 .26 /
3 .39 /
4 /
5 /
6 .68 /
7 .72 .78 /
8 /
9 .94 /