Sorting

Alexandre David
B2-206

The Problem

- **Input**: a sequence of *n* numbers $\langle a_1 \ldots a_n \rangle$.
- **Output**: a permutation (re-ordering) $\langle a_1' \ldots a_n' \rangle$ of the input sequence s.t. $a_1' \leq a_2' \ldots \leq a_n'$.
- Numbers to sort are also called **keys**.

Sorting Algorithms

- Bubble sort.
- Selection sort. n^2
- Insertion sort.
- Quick sort.
- Merge sort. $n\lg(n)$
- Heap sort.
  - Application: priority queues.

Bubble Sort

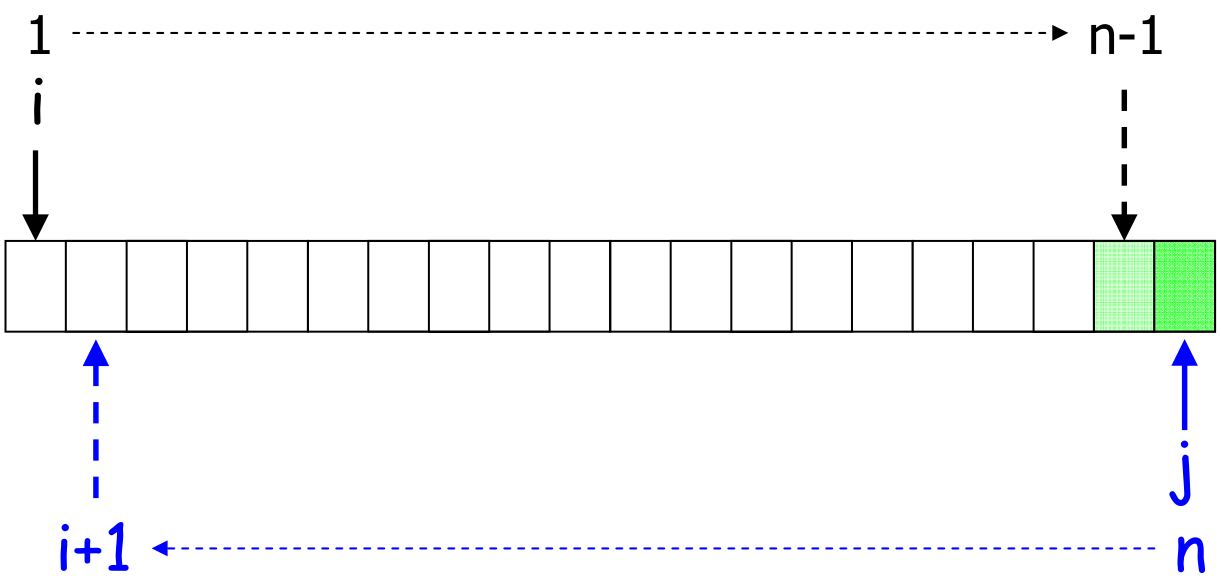
- Pseudo code in problem 2-2 p38.
  - Array indexed from 1 to n.

```
for i = 1 to n-1 do
    for j = n downto i+1 do
        if a[j] < a[j-1] then swap(a[j],a[j-1])
```

- Running time = n-1+n-2+...+1=n(n-1)/2 = $\Theta(n^2)$.

Bubble Sort

```
for i = 1 to n-1 do
    for j = n downto i+1 do
        if a[j] < a[j-1] then swap(a[j],a[j-1])
```



Bubble Sort - Correctness

- Loop invariant: Sub-array a[1..i] is sorted before the loop on i.
  - Initialization: true before 1st iteration.
  - Maintenance: If it is true before an iteration, it remains true before the next iteration.
  - Termination: The loop terminates and when it does, the invariant gives us a useful property.

Selection Sort

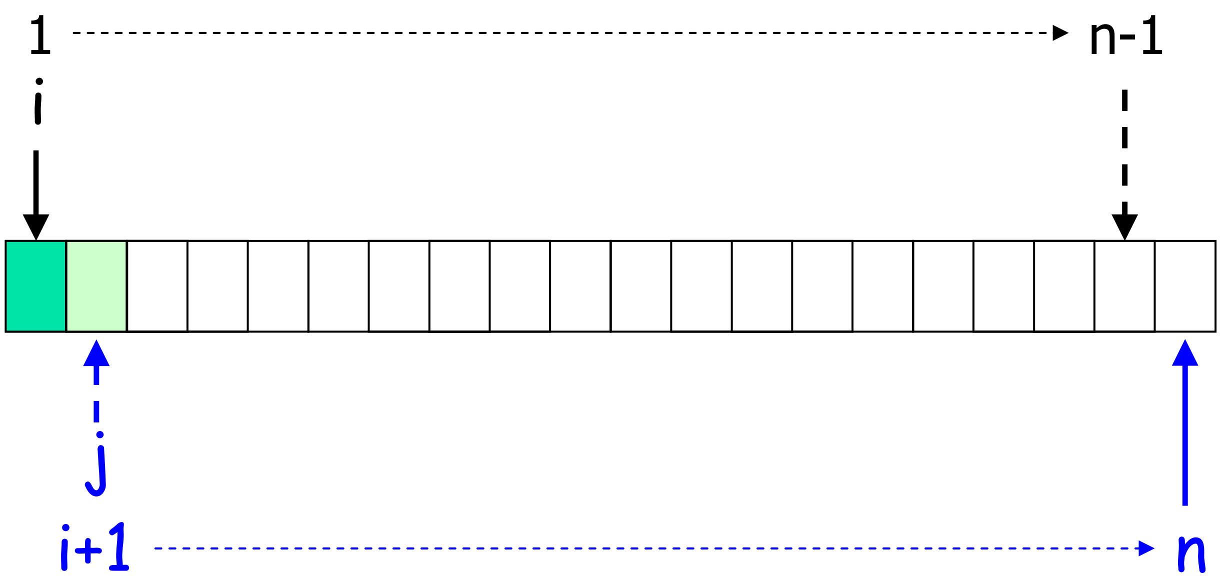
- Corresponds to exercise 2.2-2 p27.

```
for i = 1 to n-1 do
    for j = i+1 to n do
        if a[i] > a[j] then swap(a[i],a[j])
```

- Loop invariant: Sub-array a[1...i] is sorted before the loop on i.
- Running time: $n-1+n-2+\ldots+1 = n(n-1)/2 = \Theta(n^2)$.

Selection Sort

```
for i = 1 to n-1 do
    for j = i+1 to n do
        if a[i] > a[j] then swap(a[i],a[j])
```



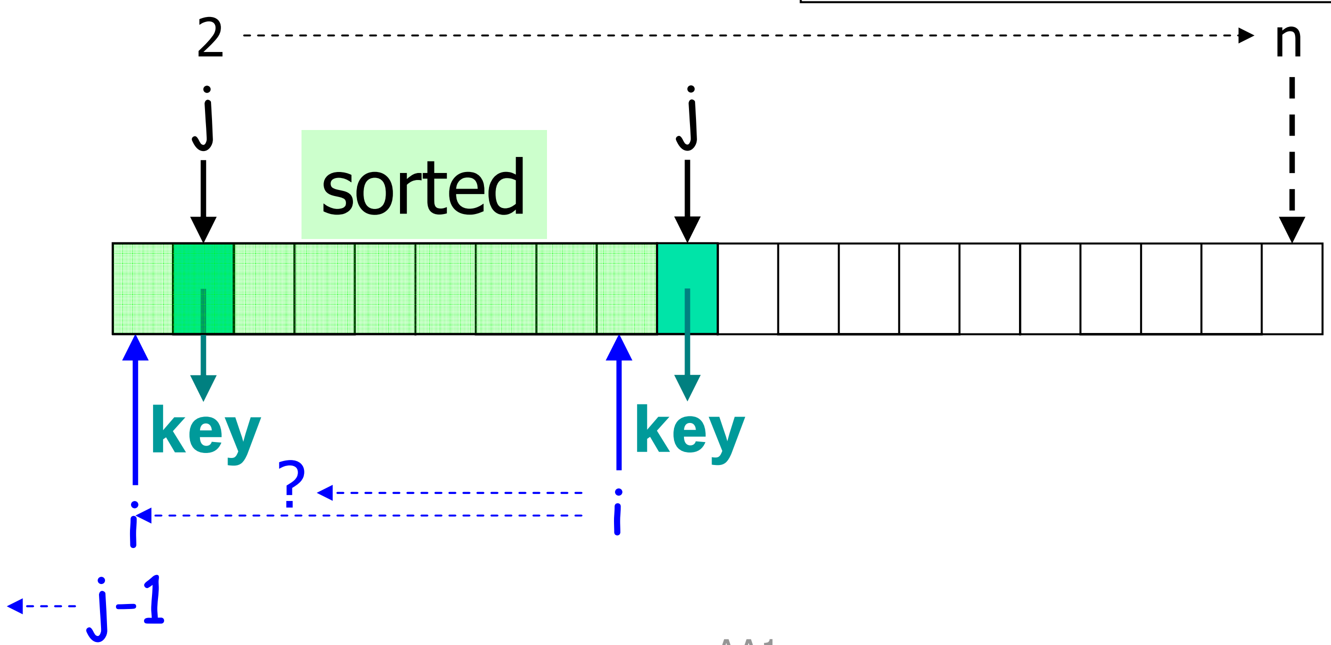
Insertion Sort

- Idea: Like when you play cards.
- Loop invariant: Sub-array a[1…j-1] is sorted before the loop on j.
- Running time: $\Theta(n^2)$?

```
for j = 2 to n do
  key = a[j]
  i = j-1
  while i > 0 and a[i] > key do
    a[i+1] = a[i]
    i = i-1
  done
  a[i+1] = key
done
```

Insertion Sort

```
for j = 2 to n do
  key = a[j]
  i = j-1
  while i > 0 and a[i] > key do
    a[i+1] = a[i]
    i = i-1
  done
  a[i+1] = key
done
```



Example

8	2	4	9	3	6
2	8	4	9	3	6
2	4	8	9	3	6
2	4	8	9	3	6
2	3	4	8	9	6
2	3	4	6	8	9

Analysis of Insertion Sort

- Count executions of loops:
  - Test n times, execute n-1 times (outer loop).
  - Test t_j times, execute t_j-1 times (inner loop).
  - $c_1 n + c_2 \sum_j t_j$
- Analysis:
  - Best case: *t_j=1, T(n)=Ω(n)*. Can be good!
  - Worst case: *t_j=j, T(n)=O(n^2)*.
  - Average case: *t_j=j/2, E[T(n)]=O(n^2)*.

Divide-and-Conquer

- *Divide-and-conquer* approach.
  - Recursive algorithms.
  - **Divide** main problem into sub-problems.
  - **Conquer** the sub-problems (solve recursively).
  - **Combine** the solutions of the sub-problems.
- Previous approaches were *incremental*.

Merge Sort

- Merge sort:
  - Divide array n into 2 arrays of size n/2.
  - Sort 2 smaller arrays with merge sort.
  - Combine the smaller arrays.
- Careful: Running time comes from
  - number of divide (= recursive calls)
  - + time of combining solutions.

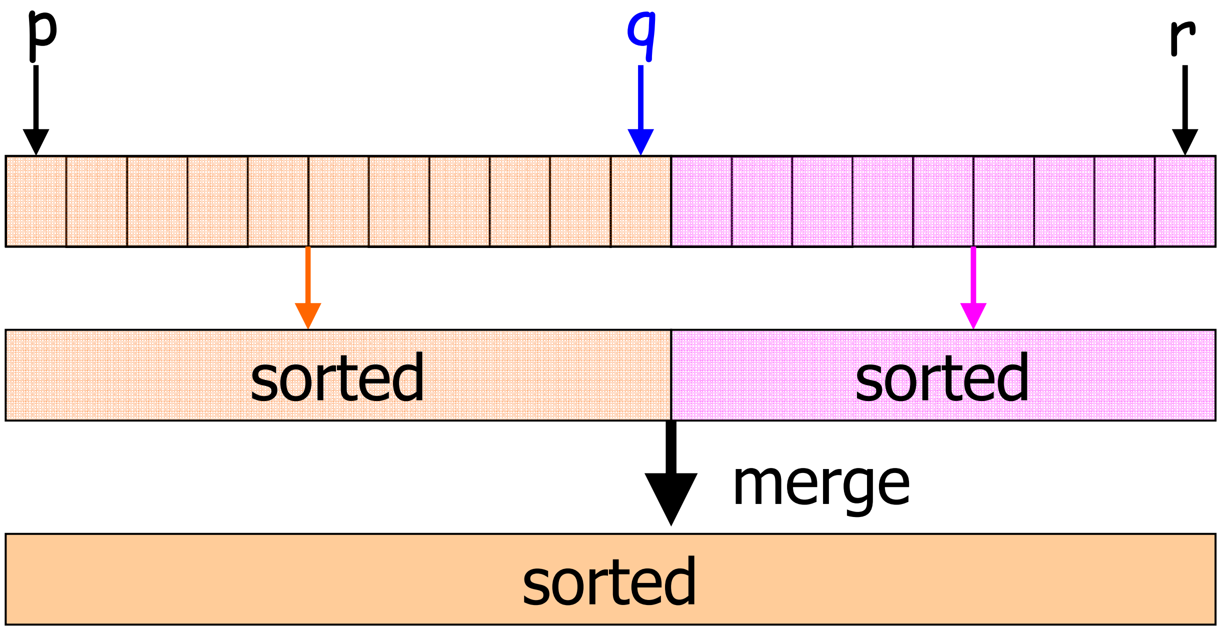
Merge Sort

- Key of the algorithm: Merge the sorted sub-arrays A[p…q] and A[q+1…r] **linearly** in time.
- Description (book) uses copies and sentinel values (to simplify tests).

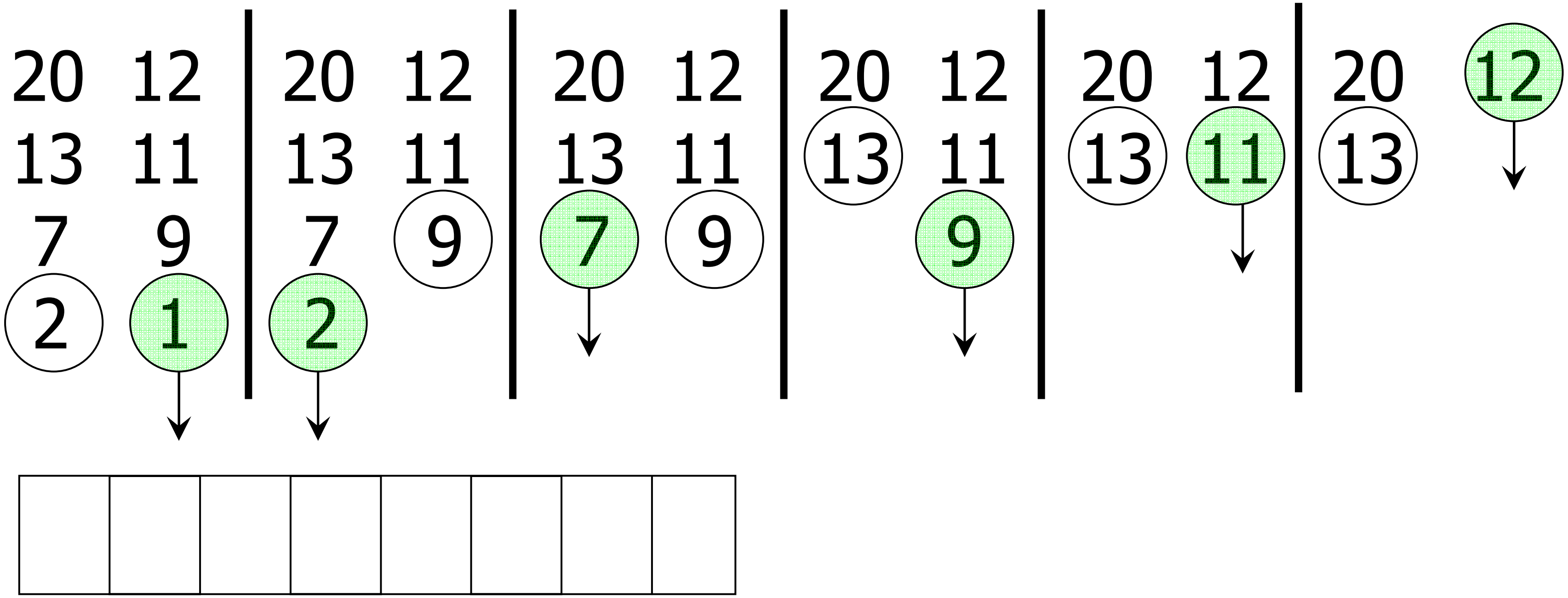
```
merge_sort(a,p,r):
  if p < r then
    q = (p+r)/2
    merge_sort(a, p, q)
    merge_sort(a, q+1, r)
    merge(a, p, q, r)
  fi
```

Merge Sort

```
merge_sort(a,p,r):
  if p < r then
    q = (p+r)/2
    merge_sort(a, p, q)
    merge_sort(a, q+1, r)
    merge(a, p, q, r)
  fi
```



Key: Merging 2 Sorted Arrays



Time = $\Theta(n)$ to merge a total of n elements (linear time).

Analyzing Merge Sort

	T(n)	merge_sort(a,p,r):
	Θ(1)	if p < r then
		q = (p+r)/2
(sloppy)	2T(n/2)	merge_sort(a, p, q)
		merge_sort(a, q+1, r)
	Θ(n)	merge(a, p, q, r)
		fi

Analyzing Merge Sort

- Running time $T(n)$ given as a recurrence. In general, for divide-and-conquer algorithms:
  - $T(n)=\Theta(1)$ if $n \leq c$ (c small enough) otherwise
  - $T(n)=aT(n/b)+D(n)+C(n)$. (divide-conquer & combine)
- Easy to solve!
- Merge-sort: $c=1$, $D(n)=1$, $C(n)=\Theta(n)$.
  - Master method: $a=2$, $b=2 \Rightarrow T(n)=\Theta(n \lg n)$.

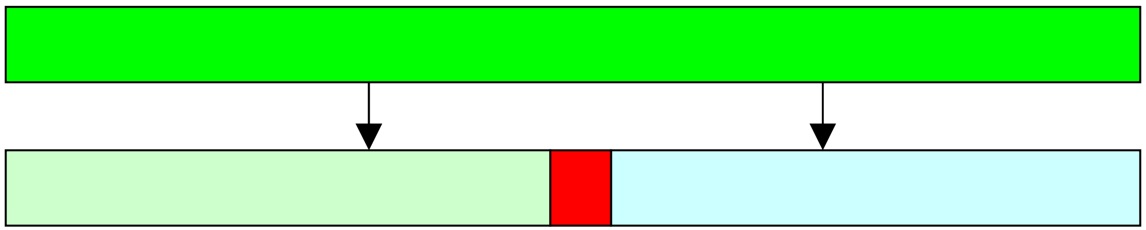
BUT: Algorithm is not in-place.

Quick Sort

- Divide-and-conquer algorithm.
  - Worst case $\Theta(n^2)$.
  - Average (robust) case $\Theta(n \lg n)$.
  - Very efficient in practice.
- In-place algorithm with small constants (complexity). Fastest sort in practice except for "almost-sorted" inputs.

Quick Sort

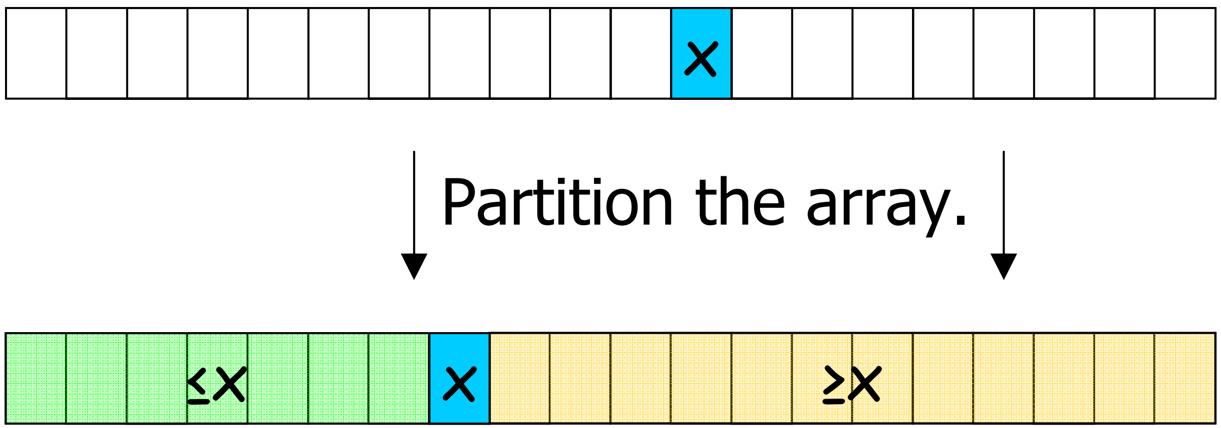
- Divide: **Partition** A[p..r] into A'[p..q-1] and A'[q+1..r] s.t. $a'_{p..q-1} \le a'_q \le a'_{q+1..r}$.



- Conquer: Sort A'[p..q-1] and A'[q+1..r] by recursive calls to quick-sort.
- Combine: A[p..r] is sorted. Nothing to do!
- Key: How to partition. We can choose a'_q arbitrarily.

Divide-and-Conquer

1)Divide: Choose pivot element.



2)Conquer: Recursive calls.
3)Combine: Trivial.

KEY: Linear time partitioning sub-routine.

Quick Sort Partitioning

pivot →

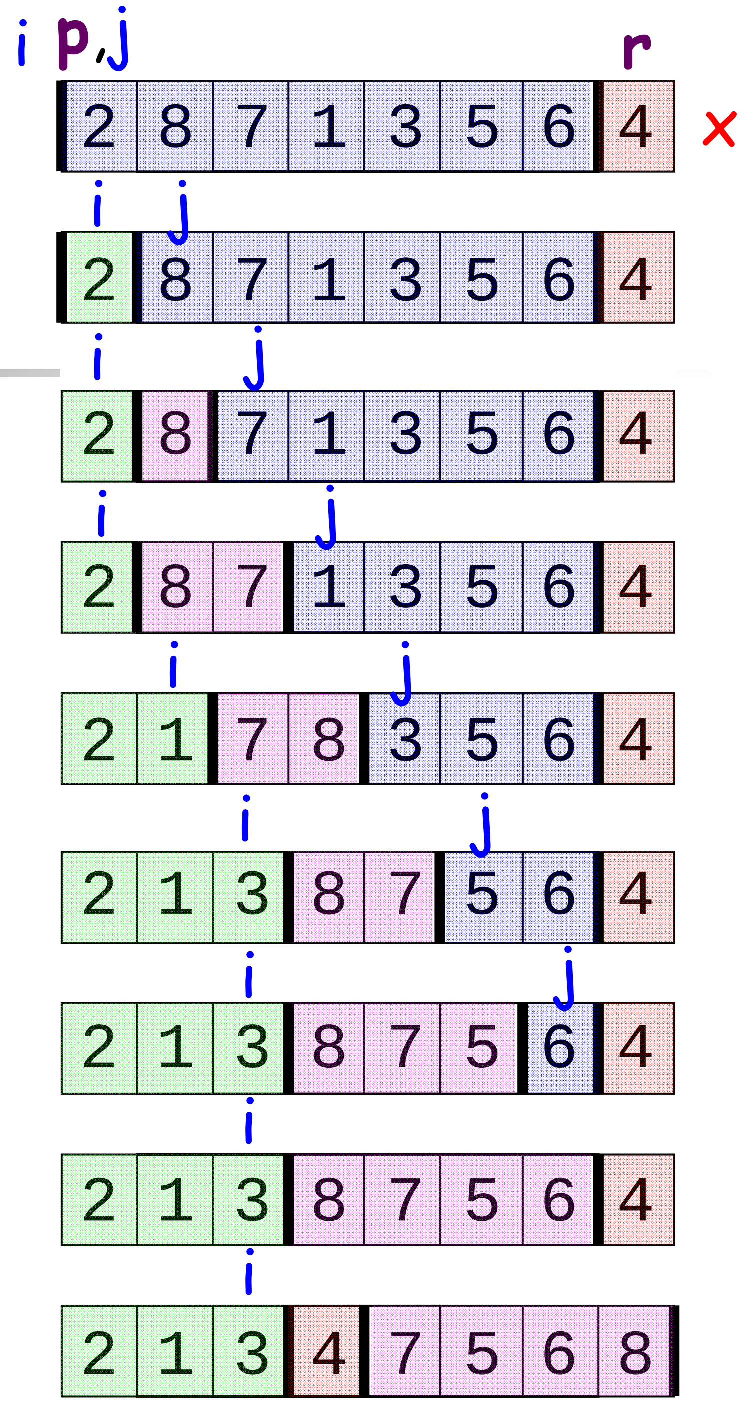
```
partition(A,p,r)
x = A[r]
i = p-1
for j = p to r-1 do
   if A[j] ≤ x then
      i = i+1
      swap(A[i],A[j])
   fi
done
swap(A[i+1],A[r])
return i+1
```

- Equivalent to selection-sort for worst case.

Variants

- Hoare's partitioning more efficient.
  - Problem 7-1 p160.
  - Fewer swaps in practice.
- Randomized quick-sort:
  - Choose the pivot randomly.
  - No worst-case input but unlucky run may take $\Theta(n^2)$.
  - More robust.
- Combine quick-sort with insertion sort.

```
partition(A,p,r)
x = A[r]
i = p-1
for j = p to r-1 do
   if A[j] ≤ x then
      i = i+1
      swap(A[i],A[j])
   fi
done
swap(A[i+1],A[r])
return i+1
```



≤x	>x	?	x

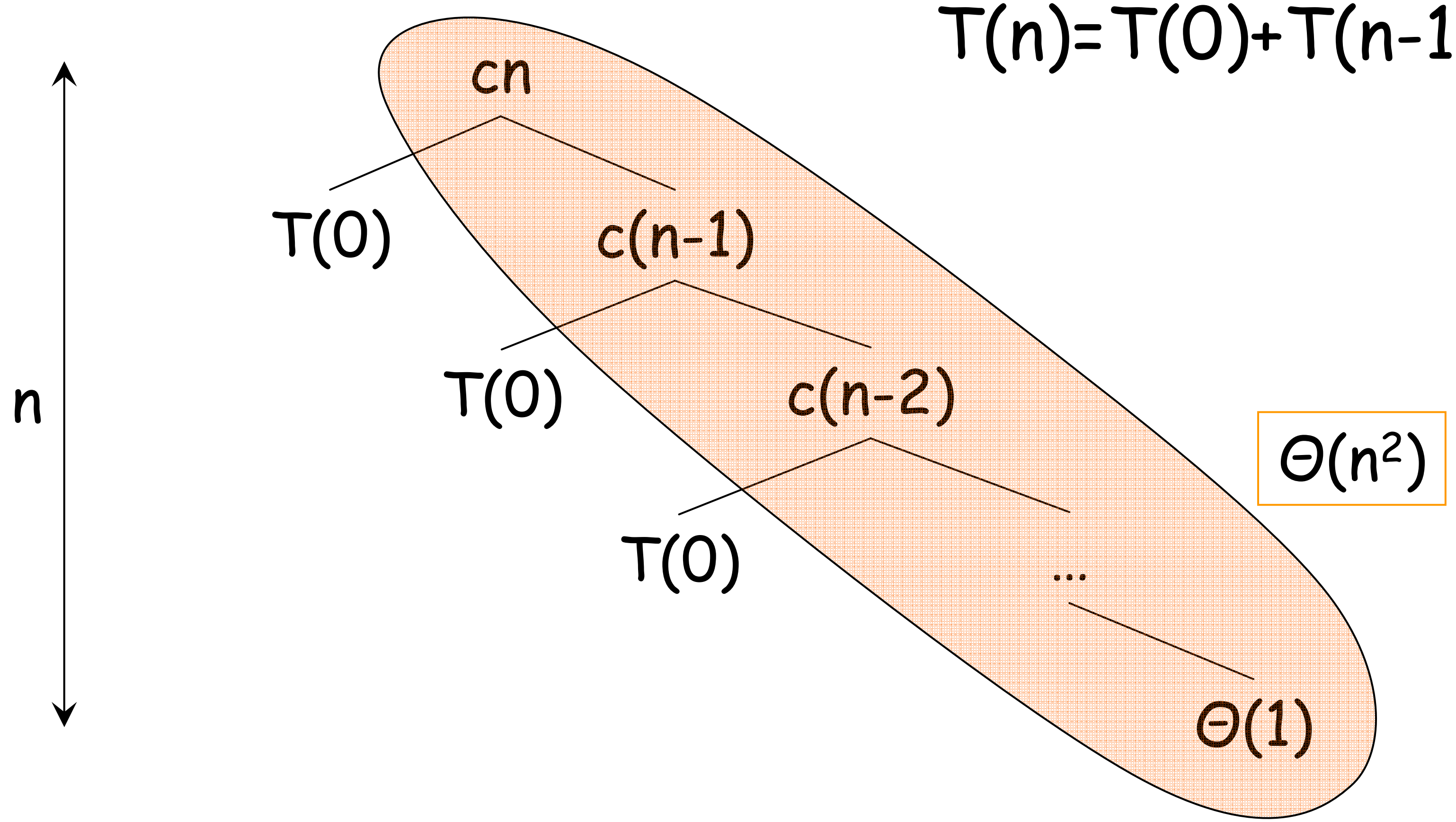
Worst Case of Quick Sort

- Input sorted or reverse order.
- Partition around *min* or *max* element.
- Always no element for one side.

$$
\begin{aligned}
T(n) &= T(0) + T(n-1) + \Theta(n) \\
&= \Theta(1) + T(n-1) + \Theta(n) \\
&= T(n-1) + \Theta(n) \\
&= \Theta(n^2)
\end{aligned}
$$

Worst Case Recursion Tree

$$T(n) = T(0) + T(n-1) + cn$$



Best Case Analysis
Intuition

- If we are lucky, the partition splits the array evenly. Like merge-sort:

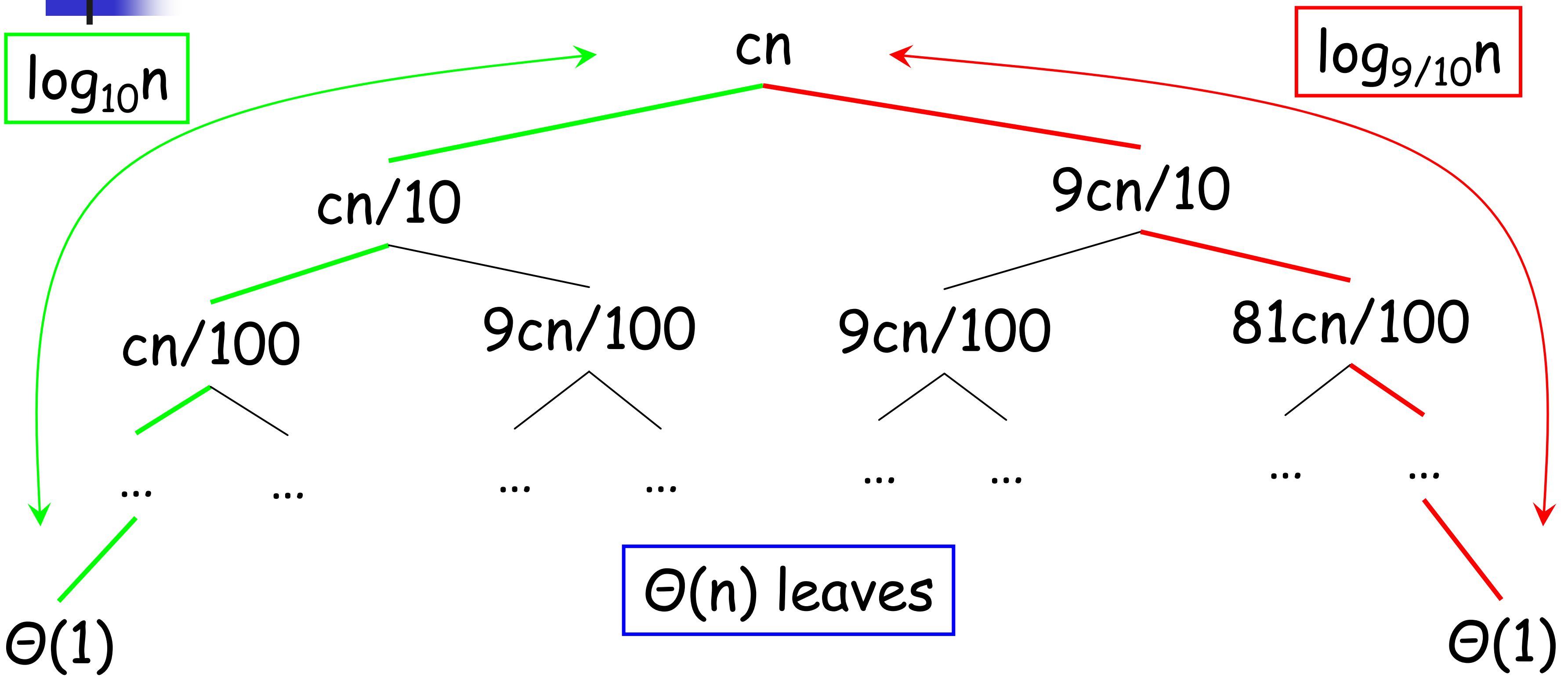
$$T(n) = 2T(n/2) + \Theta(n)$$
$$= \Theta(n \lg n)$$

- What if the split is always 1/10 : 9/10?

$$T(n) = T(n/10) + T(9n/10) + \Theta(n)$$

? Solution?

Recursion Tree 1/10 : 9/10



$cn\log_{10}n+\Theta(n) \le T(n) \le cn\log_{10/9}n+\Theta(n) \longrightarrow \Theta(n\lg n)$ *lucky*

Performance of Quick Sort

- Depends on the input (balanced array or not) and the choice of the pivot.
- Worst-case (sorted, reversed, equal elements): $\Theta(n^2)$.
- Best-case: (balanced): $T(n)=\Theta(n \lg n)$.
- Average case?
  - Remark: 1/10 : 9/10 still in $n \lg n$.
  - Average is $\Theta(n \lg n)$.
  - Analysis: Randomized quick-sort…

Randomized Quick Sort

- Pick up a random pivot for the partitioning.
- Running time unpredictable.
- No assumption on the input distribution.
- No specific input gives the worst case.
- Used for analysis of quick sort for the average case.

Problem 7-2

Randomized Quick Sort Analysis

- T(n)=random variable for the running time of randomized quick sort (input of size n), assuming independent random numbers.
- For k=0…n-1, define the indicator random variable

$$X_k = \begin{cases} 1 & \text{if partition generates a } k : n\text{-}k\text{-}1 \text{ split} \\ 0 & \text{otherwise} \end{cases}$$

$E[X_k]=\Pr\{X_k=1\}=1/n$ since all splits are equally likely, assuming distinct elements.

T(n) with Indicator Variable

$$T(n) = \begin{cases} T(0) + T(n-1) + \Theta(n) & \textit{if } 0 : n-1 \textit{ split}, \\ T(1) + T(n-2) + \Theta(n) & \textit{if } 1 : n-2 \textit{ split}, \\ \vdots & \\ T(n-1) + T(0) + \Theta(n) & \textit{if } n-1 : 0 \textit{ split}. \end{cases}$$

$$T(n) = \sum_{k=0}^{n-1} X_k \left(T(k) + T(n-k-1) + \Theta(n) \right)$$

Expectation

$$E[T(n)] = E\left[\sum_{k=0}^{n-1} X_k\left(T(k)+T(n-k-1)+\Theta(n)\right)\right]$$

$$= \sum_{k=0}^{n-1} E\left[X_k\left(T(k)+T(n-k-1)+\Theta(n)\right)\right]$$

$$= \sum_{k=0}^{n-1} E[X_k]\cdot E[T(k)+T(n-k-1)+\Theta(n)]$$

$$= \frac{1}{n}\sum_{k=0}^{n-1} E[T(k)] + \frac{1}{n}\sum_{k=0}^{n-1} E[T(n-k-1)] + \frac{1}{n}\sum_{k=0}^{n-1} \Theta(n)$$

$$= \frac{2}{n}\sum_{k=0}^{n-1} E[T(k)] + \Theta(n)$$

Tough recurrence!

Solving the Recurrence

$$E[T(n)] = \frac{2}{n}\sum_{k=2}^{n-1} E[T(k)] + \Theta(n)$$

(k=0 and k=1 can be absorbed in Θ(n))

- **Prove** *E[T(n)]≤an* lg*n* for a constant a>0. Choose a large enough so that *an* lg*n* dominates *E[T(n)]* for sufficiently small *n≥2*.
- Use fact: $\sum_{k=2}^{n-1} k \lg k \le \frac{1}{2} n^2 \lg n - \frac{1}{8} n^2$ (exercise)

Substitution Method…

$$E[T(n)] \le \frac{2}{n}\sum_{k=2}^{n-1} ak\lg k + \Theta(n)$$ (use substitution)

$$= \frac{2a}{n}\left(\frac{1}{2}n^2\lg n - \frac{1}{8}n^2\right) + \Theta(n)$$ (use fact)

$$= an\lg n - \left(\frac{an}{4} - \Theta(n)\right)$$

$$\le an\lg n$$ with *a* large enough

$\boxed{E[T(n)]=O(n\lg n)}$