Algorithms & Architecture Representing and Manipulating Numbers

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B2-206

What this lecture is about: Secrets of number encoding, how to hack integers (and floats). ☺

Today

- Representation of numbers – introduction.
- Integer coding.
- Basic arithmetic.
- Hexadecimal notations.
- Endianness.
- IEEE floats.
- Implementation of functions.
- Numerical precision.

Basics

Natural/Real Numbers

- Base 10
- Infinite
- Exact

Computer Numbers

- Base 2
- Finite
- Rounding - overflow

In this lecture

- How to represent numbers – range, encoding.
- Arithmetic.
- How to use these numbers.

Unit for coding = bit.

Finite number of bits, finite numbers.

Signed/unsigned: similar representations but different interpretations.

Questions

- ? How to code negative numbers?
- ? How to code real numbers?
- ? Which kind of precision do we get?
  - Small numbers vs. big numbers.

Example

- Overflow:

```
main() {
  printf("%d\n",200*300*400*500);
}
```

outputs -88490188. ?

- Fix – sort of:

```
main() {
  printf("%lld\n",200LL*300LL*400LL*500LL);
}
```

outputs 120000000000. ? What if I forget LL?

Limits of int32_t: -2^31…2^31-1.

Example

- Loss of precision:
  (3.14+1e20)-1e20==0.0
  3.14+(1e20-1e20)==3.14
- ? Test x == 0.0 not very useful when solving equations.

Information Storage

- Basic unit is the byte (=8 bits).

C-declaration	Typical 32-bit	Compaq Alpha
char	1	1
short int	2	2
int	4	4
long int	4	8
char *	4	8
float	4	4
double	8	8

Size of the different types of integers depends on the architecture. Addressing is limited by the size of pointers that gives the size of addressable memory.

Features

- Limits on addressable memory.
- Size of registers – 32/64.
- Aligned memory allocation (32/64 bits).
- Careful on addressing:

```
main() {
  char a[]="Hello world!";
  int *p=&a[1];
  printf("%d\n",*p);
}
```

?

Integer Coding

- Unsigned integers: $UB = \sum_{i=0}^{w-1} x_i 2^i$
- Signed integers: $SB = -x_{w-1} 2^{w-1} + \sum_{i=0}^{w-2} x_i 2^i$
  Called 2 complement.

w: size of a word (in bits)
x: bits (0 or 1)

- Highest bit codes the sign.

Basic Arithmetic

- Logical operations (bitwise): &,|,^,~,<<,>>.
  Example: a ^= b; b ^= a; a ^= b; ?
- Arithmetic operations: + - * /. ?
- Careful with shifts on signed integers!
- Do not mess up with boolean operations (&&, ||).

Properties

- Operations are the same on signed/unsigned integers.
- Check properties on the notes.
- Operations based on the algebra $<Z_n,+_n,*_n,-_n,0,1>$. Operations modulo n and -a=0 or -a=n-a.

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Basic properties:

•commutativity: a+b=b+a, a*b=b*a

•associativity: (a+b)+c=a+(b+c), (a*b)*c=a*(b*c)

•distributivity: a*(b+c)=a*b+b*c

•identities: a+0=a, a*1=a

•annihilator: a*0=0

•cancellation: -(-a)=a

Properties for integer ring <Z,+,*,-,0,1> and for boolean algebra <{0,1},|,&,~,0,1> are similar (| instead of +, & instead of *).

Unique for integer ring: a+-a=0.

Unique for boolean algebra:

•distributivity: a|(b&c) = a&(b|c)

•complement: a|~a=1, a&~a=0

•idempotency: a&a=0, a|a=a

•absorption: a|(a&b)=a, a&(a|b)=a

•DeMorgan laws: ~(a&b)=~a|~b, ~(a|b)=~a&~b

Shifts & Masks

- Read bit n: Use mask (1 << n).
- Set bit n on int bits[]:
  ```
  ipos   = n / 32;
  imask = 1 << (n % 32);
  bits[ipos] |= imask;
  ```
- Shifts as division/multiplication by powers of 2.
- Shifts for negative numbers has "wrong" rounding.
  Correct: (x<0?(x+(1<<k)-1):x)>>k.

How does it work?

```
  1011
 +1101
  111
 11000
```

Subtraction?

```
    1011   multiplicand
   *1101   multiplier
    1011
   0000    partial
  1011     products
 1011
10001111   product
```

How does it work?

10010011/1011=?

```
1011 |10010011
divisor   dividend
```

quotient

```
        0
1011 |10010011
      1011
```

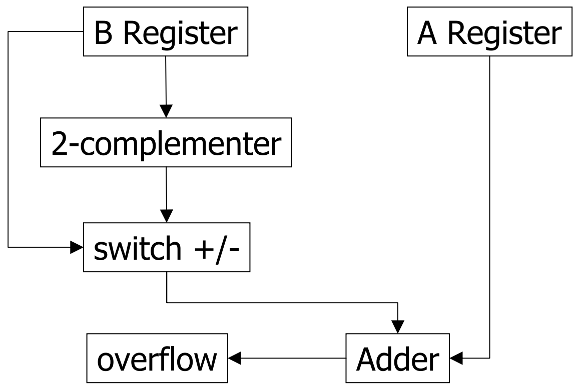
```
        00
1011 |10010011
       1011
```

```
       000
1011 |10010011
      1011
```

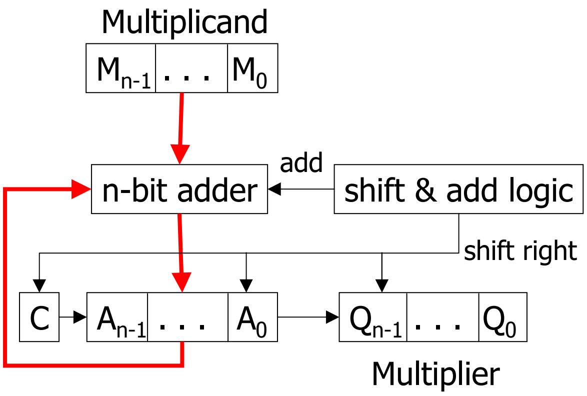
```
       0000
1011 |10010011
       1011
```

```
       000011
1011 |10010011
      -1011↓
       001110  partial remainder
          1011
```

Block Diagram for A+B, A-B



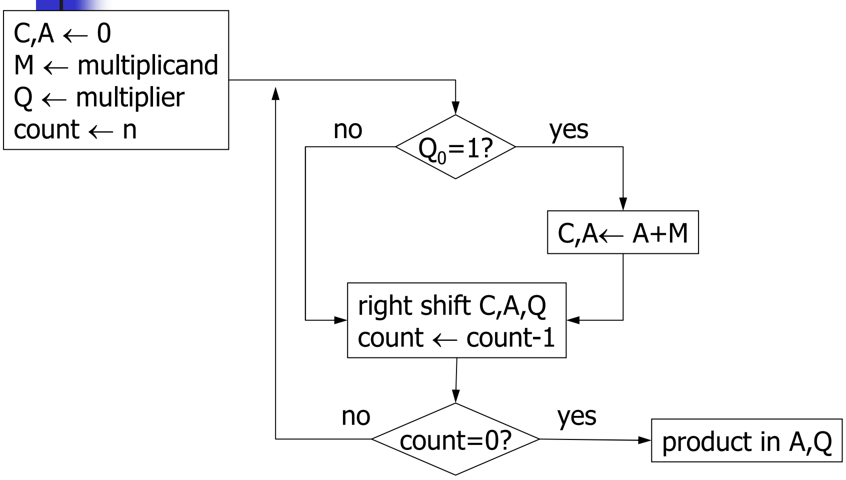
Block Diagram for M*Q



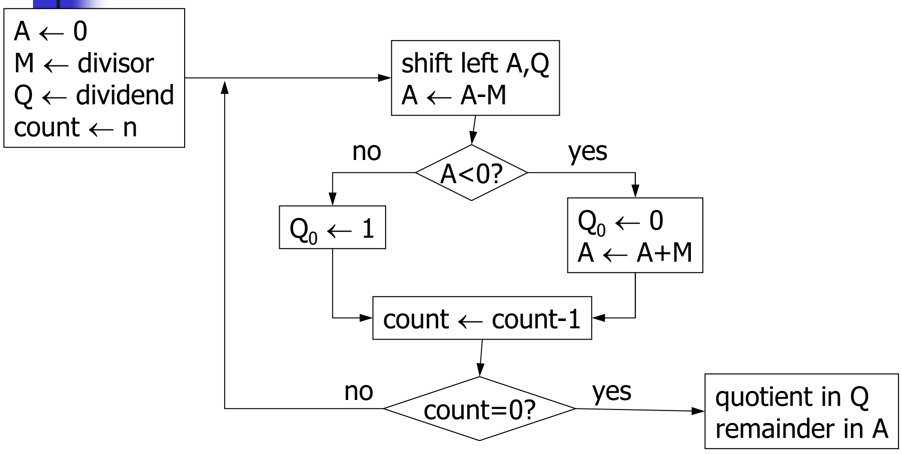

```
C  A    Q    M
0 0000 1101 1011

0 1011 1101 1011 +
0 0101 1110 1011 >>

0 0010 1111 1011 >>

0 1101 1111 1011 +
0 0110 1111 1011 >>

1 0001 1111 1011 +
0 1000 1111 1011 >>
```

Product in A,Q

Flowchart for unsigned *



Flowchart for unsigned /



Arithmetic

- Machine code of + - * / same for int/uint.
- Integer convertion == type casting.
  - Padding for the sign (int).
  - Conversion is modulo the size of the new int.
  - Beware of implicit conversions in C!
- Optimizations for some operations:

```
2*a    == a+a == a<<1      a/2    == a>>1
a*2^i == x << i            a/2^i == a>>i
a%2^i == a & ((1<<i)-1)    2^i    == 1<<i
```

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Previous read example for a positive index n:

```
ipos = n >> 5;
imask = 1 << (n & 31);
bits[n] |= imask;
```

Notes

- Beware of precedence of operators:
  - if (x & mask == value) WRONG
  - if ((x & mask) = value) RIGHT
- Test odd numbers: if (x & 1)
- Careful:

```
unsigned int i;
for(i = 0; i < n-1; ++i) …
```

?

Hexadecimal Notation

- Learn the first powers of 2.
- Hexadecimal more useful:
  - One digit codes 4 bits. 0…F=0…15=16 numbers.
  - C notation: 0x…
- Examples:
  - 0xa57e=1010 0101 0111 1110
    10=8+2,5=4+1,7=4+2+1,14=8+4+2
  - 0xf = 1111, 0x7=0111, 0x3=0011

Remember: Individual bits are accessed by shifts and masks.

Ranges:

•uint: 0…0xffffffff

•int: 0x80000000…0x7fffffff

0xf, 0x7, 0x3, 0x7f, … are strings of consecutive 1s.

To get the encoding of negative numbers, use -a=~a+1.

Hack

- Allocated memory is 32 bits aligned
  - 2 lower bits are = 0.
  - Use them to store data.
  - But it's a hack.

Endianness: Beware!

- "0xa57e" is a notation for humans. Corresponds to "1010 0101 0111 1110" in base 2.
  - Little endian: stored as 0111111010100101.
  - Big endian: stored as 1010010101111110.
- Does not matter in C, except for
  - bitmap manipulation
  - device drivers
  - network transfers

Testing for endianness

- Write a value on 32 bits.
  - Read 8/16 bits and check what was written.
  - Exercise for Sun/Intel.

```
main() {
    int a = 0xf0000000;
    char *c = &a;
    printf("%x\n", *c);
}
```

- ?
  - Intel?   0
  - Sun?   fffffff0
  - What if a = 0x70000000?

•endianness

•sign coding

•integer convertion

Representation of reals

- How to code a real number with bits?
  - Finite precision $\rightarrow$ approximation.
  - Represent very small and very large numbers $\rightarrow$ density of encoding varies.
- Scientific notation used, e.g. (base 10), 3.141e2 – but in base 2.
- Starter: fractional numbers – bad for large or small numbers.
  - Decimal (d): $d = \sum_{i=-n}^{m} 10^i d_i$
  - Binary(b): $b = \sum_{i=-n}^{m} 2^i b_i$

IEEE floats

- IEEE floating point standard
  - $V=(-1)^S M*2^E$
  - Number of bits (float/double):
    s[1], m[23/52], e[8/11].
  - Normalized and de-normalized values.
  - Bit fields: s, m, e to code respectively S, M, E.
- Normalized values ($e \neq 0$, $e \neq 111\ldots$)
  - E=e-bias (-126…127/-1022…1023).
  - M=1+m   ($1 \leq M < 2$)
  - Trick for more precision: implied leading 1

Bias is used for a smooth transition between normalized and de-normalized numbers. IEEE distinguishes between +0.0 and -0.0: one more reason to test with a tolerance.

NaN generated for √-1, inf-inf, 0/0.

IEEE floats

- De-normalized values (e= 0 or 11…)
  - e=0:   E=1-bias, bias=2^{k-1}-1
    - Coding compensates for M not having an implied leading 1.
    - M=m
    - For numbers very close to 0.
  - e=11…:
    - m=0, (signed infinite)
    - m!=0, NaN.

Features of IEEE floats

- +0.0 == 0 (binary representation = 00…).
- If interpreted as unsigned int, floats can be sorted (+x ascending, -x descending).
- All int values representable by doubles.
- ? Not all int values representable by floats.
  - round to even (avoid stat. bias)
  - round towards 0
  - round up
  - round down
  - can't choose in C…

Properties

- Operations NOT associative.
- Not always inverse (infinity).
- Loss of precision.
- Ex: x=a+b+c; y=b+c+d;
  Optimize or not?
- Monotonicity $a \geq b \Rightarrow a+x \geq b+x$
- Casts:
  - int2float rounded, double2float rounded/overflow
  - int2double, float2double OK
  - float2int, double2int truncated/rounded/overflow.

Important for compilers and programmers.

IA32: The good and the bad

- Good: Uses internally 80 bits extended registers for more precision.
- Bad:
  - Stack based.
  - Side effects like changing values when loading or saving numbers in memory whereas register transfers are exact.
- Extensions: MMX, SSE, Altivec. SIMD instructions = operations working in parallel on multiple data.

Implementation - summary

- Integers
  - addition/subtraction simple
  - multiplication:
    `r=0; while(b) do { if (b&1) r+=a; a+=a; b>>=1; }`
  - division iterative
- Floats
  - addition/subtraction complicated
    - check for 0, align significands, add/sub, normalize
    - guard bits to avoid losing precision on very close numbers (1.00…*2^1 – 1.11…2*0)
  - multiplication/division principle simpler
    - multiply/divide significands, add/sub exponents, detect over/under-flow.

Complex functions

- Lagrange polynomials
- Taylor series

$$P_n(x) = \sum_{i=0}^{n} f(x_i) \prod_{k=0,k\neq i}^{n} \frac{x - x_k}{x_i - x_k}$$

$$f(x) = \sum_{i=0}^{+\infty} f^{(i)}(x_0) \frac{(x - x_0)^i}{i!}$$

- Other numerical methods that converge rapidly.
- Intel:: cos, sin, sqrt, in hardware.
- Special (int): random generator $x_{i+1}=(ax_i+c)\%m$.
  - Simple but correlation between successive values.
  - Higher bits better quality than lower bits.

Numerical precision

- Evaluation of precision
  - absolute $x \pm \alpha$
  - relative $x*(1 \pm \alpha)$
- Be careful with division by very small values: Can amplify numerical errors.
  - Numerical justification for Gauss' method to solve linear equations.
- More in courses on numerical methods.