AA1 - Introduction & Growth of Functions

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1 Euclid's Algorithm

1 Prove the correctness of the algorithm. Hint: Do it in two steps, (1) if n divides m then *gcd(m,n)=n*, and (2), *gcd(m,n)=gcd(n, m mod n)*.

Suggested solution (1) $gcd(m,n)$ divides n implies $gcd(m,n) \leq n$. n divides n and m implies $n \leq gcd(m,n)$. Therefore n divides m implies $gcd(m,n) = n$. (2) Let's prove that if $m = nt+r$, for integers t and r, then $gcd(m,n) = gcd(n,r)$. Every common divisor of m and n also divides r ($m = ap$ and $n = aq$ give $r = a(p - qt)$), in particular $gcd(m,n)$ divides r too. So $gcd(m,n) \leq gcd(n,r)$. The reverse is also true because every divisor of n and r also divides m, which yields $gcd(n,r) \leq gcd(m,n)$. (3) What (1) and (2) show is the invariant of the loop. The loop terminates when we have $gcd(n,0) = n$, which happens since n is monotonically decreasing.

2 Euclid's algorithm, as presented in Euclid's treatise, uses subtractions rather than integer divisions. Write a pseudocode for this version of Euclid's algorithm.

Suggested solution Use $gcd(m,n) = gcd(n, m - n)$. You can prove it for fun.

3 - For fun Euclid's game starts with two unequal positive numbers on the board. Two players move in turn. On each move, a player has to write on the board a positive number equal to the difference of two numbers already on the board; this number must be new, i.e., different from all the numbers already on the board. The player who cannot move loses the game. Should you choose to move first or second in this game?

Suggested solution Use Euclid's algorithm based on substractions. If $gcd(m,n) = 1$, it means 1 will be generated, i.e., all the numbers will be generated from 1 to m. Let's say m is max, if m is odd, play first, if m is even, play second. If $gcd(m,n) = p$, there is a cycle. Let's say $ap = m$, a is the number of numbers at the end on the board. If a is odd play first, if a is even play second.

2 Algorithm Problem Solving

1 Design a simple algorithm for the string-matching problem.

Suggested solution Trivial.

2 Icosian game: a century after Euler's discovery, another famous puzzle – invented by the renown Irish mathematician Sir William Hamilton (1805-1865) – was presented to the world under the name of the Icosian Game. The game was played on a circular wodden board on which the graph of Fig.1 was carved. Find a *Hamiltonian circuit* – a path that visits all the graph's vertices exactly once before returning to the starting vertex – for this graph.

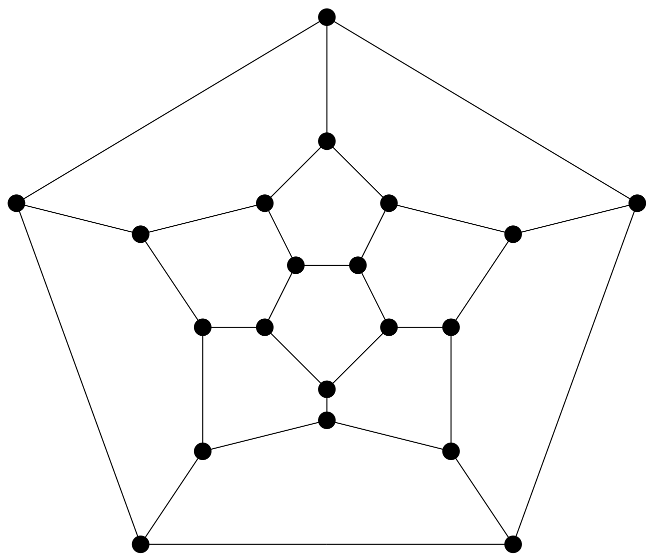

Figure 1: Icosian game.

3 Design an algorithm for the following problem: Given a set of n points in the $x - y$ coordinate plane, determine whether all of them lie on the same circumference.

Suggested solution Choose 3 points, compute the center of the unique circle and the radius of the circle. Check that all the other points are at this radius from the center.

4 For each function $f(n)$ and time t in the following table, determine the largest size n of a problem that can be solved in time t, assuming that the algorithm to solve the problem takes $f(n)$ microseconds. Dispatch the work within your group so you don't do everything yourself.

	1 sec	1 min	1 hour	1 day	1 month	1 year	1 century
$lg(n)$							
$\sqrt{n}$							
n							
$n * lg(n)$							
n^2							
n^3							
2^n							
$n!$							

Suggested solution

	1 sec	1 min	1 hour	1 day	1 month	1 year	1 century
$lg(n)$	e^{10^6}	e^{6*10^7}	$e^{3.6*10^9}$	$e^{8.6*10^{10}}$	$e^{2.5*10^{12}}$	$e^{3.1*10^{13}}$	$e^{3.1*10^{15}}$
$\sqrt{n}$	1e12	3.6e15	1.3e19	7.5e21	6.7e24	1e27	1e31
n	1e6	6e7	3.6e9	8.6e10	2.5e12	3.1e13	3.1e15
$n * lg(n)$	8.7e4	4e6	1.9e8	3.9e9	1e11	1.1e12	1e14
n^2	1e3	7.7e3	6e4	2.9e5	1.6e5	5.6e6	5.6e7
n^3	100	391	1532	4420	13736	31601	1446679
2^n	20	25	31	36	41	44	51
$n!$	9	10	12	13	15	16	17

3 Numerical Example

Let us compute $q = a/b$ using Newton-Raphson's method. Find an algorithm to compute q by searching the solution of $f(x) = 0$ for $f(x) = 1/x - b$.

Suggested solution Trivial.

4 Asymptotic Notations

1 Let $f(n)$ and $g(n)$ be asymptotically non-negative functions. Using the basic definition of Θ-notation, prove that $max(f(n), g(n)) \in \Theta(f(n) + g(n))$.

Suggested solution $f(n) + g(n) \geq max(f(n), g(n)) \implies max(f(n), g(n)) \in O(f(n) + g(n))$. $max(f(n), g(n)) \geq \frac{f(n)+g(n)}{2} \implies max(f(n), g(n)) \in o(f(n) + g(n))$. Therefore $max(f(n), g(n)) \in \Theta(f(n) + g(n))$.

2 Show that for any real constants a and b, where $b > 0$, $(n + a)^b = \Theta(n^b)$.

Suggested solution $(n+a)^b = \sum_{i=0}^{i=b} \binom{i}{b} n^i a^{b-i}$. We isolate n^b: $(n+a)^b = n^b + \sum_{i=0}^{i=b-1} \binom{i}{b} n^i a^{b-i}$. The second term is a polynomial in n^{b-1}. Therefore it can be upper-bounded but a cn^b, c being a constant dependent on the constants of the polynomial. We obtain $(n+a)^b \in O(n^b)$. The same holds for the lower-bound and we have $(n+a)^b \in o(n^b)$. Therefore $(n+a)^b \in \Theta(n^b)$.

3 Explain why the statement "The running time of algorithm A is at least $O(n^2)$ is meaningless".

Suggested solution $O(n^2)$ is an upper-bound. It is *at most* $O(n^2)$.

5 Fibonacci Numbers

Prove by induction that the ith Fibonacci number satisfies the equality

$$F_i = \frac{\phi^i - \hat{\phi}^i}{\sqrt{5}},$$

where ϕ is the golden ratio and $\hat{\phi}$ is its conjugate: $\phi = \frac{1+\sqrt{5}}{2}$ and $\hat{\phi} = \frac{1-\sqrt{5}}{2}$.

Suggested solution $F_0 = 0$ holds. $F_1 = 1$ holds. Assume that $F_k = \frac{\phi^k - \hat{\phi}^k}{\sqrt{5}}$ holds $\forall k < n$. $F_n = F_{n-1} + F_{n-2}$ by definition of F_n.
$F_n = \frac{\phi^{n-1} - \hat{\phi}^{n-1}}{\sqrt{5}} + \frac{\phi^{n-2} - \hat{\phi}^{n-2}}{\sqrt{5}}$ by induction. Let's verify an identity:

$$\begin{aligned}
\phi^{i-1} - \hat{\phi}^{i-1} + \phi^{i-2} - \hat{\phi}^{i-2} &= (1 + \tfrac{1+\sqrt{5}}{2})\phi^{i-2} - (1 + \tfrac{1-\sqrt{5}}{2})\hat{\phi}^{i-2} \\
&= \tfrac{4+2+2\sqrt{5}}{4}\phi^{i-2} - \tfrac{4+2-2\sqrt{5}}{4}\hat{\phi}^{i-2} \\
&= \tfrac{1+2\sqrt{5}+5}{4}\phi^{i-2} - \tfrac{1-2\sqrt{5}+5}{4}\hat{\phi}^{i-2} \\
&= (\tfrac{1+\sqrt{5}}{2})^2\phi^{i-2} - (\tfrac{1-\sqrt{5}}{2})^2\hat{\phi}^{i-2} \\
&= \phi^i - \hat{\phi}^2
\end{aligned}$$

We conclude that $F_n = \frac{\phi^n - \hat{\phi}^n}{\sqrt{5}}$, which is, the induction hypothesis holds for n too. By induction the equality holds for all $n \leq 0$.