

# Sorting – Solution

Alexandre David

## Proof Complement (7-2.b)

The proof uses  $\lg(2) = 1$ ,  $\lceil \frac{n}{2} \rceil \geq \frac{n}{2}$ ,  $\lceil \frac{n}{2} \rceil \leq \frac{n}{2} + 1$ , and  $\lg(x - 1) \leq \lg(x) - \frac{1}{x}$  that comes from  $\frac{1}{x} \leq \lg(x) - \lg(x - 1) \leq \frac{1}{x-1}$ .

$$\sum_{k=2}^{n-1} k \lg(k) = \sum_{k=2}^{\lceil \frac{n}{2} \rceil - 2} k \lg(k) + \sum_{k=\lceil \frac{n}{2} \rceil - 1}^{n-1} k \lg(k) \quad (1)$$

$$\leq \lg\left(\frac{n}{2}\right) - 1 \sum_{k=2}^{\lceil \frac{n}{2} \rceil - 2} k + \lg(n-1) \sum_{k=\lceil \frac{n}{2} \rceil - 1}^{n-1} k \quad (2)$$

$$\leq \left(\lg\left(\frac{n}{2}\right) - \frac{2}{n}\right) \frac{1}{2} \left(\lceil \frac{n}{2} \rceil - 3\right) \lceil \frac{n}{2} \rceil + \left(\lg(n) - \frac{1}{n}\right) \frac{1}{2} (n-1-\lceil \frac{n}{2} \rceil) (\lceil \frac{n}{2} \rceil + n - 2) \quad (3)$$

$$\leq \left(\lg\left(\frac{n}{2}\right) - \frac{2}{n}\right) \frac{1}{2} \left(\frac{n}{2} - 2\right) \left(\frac{n}{2} + 1\right) + \left(\lg(n) - \frac{1}{n}\right) \frac{1}{2} \left(\frac{n}{2} - 1\right) \left(\frac{3n}{2} - 1\right) \quad (4)$$

$$= \frac{1}{2} \left(\lg(n) - 1 - \frac{2}{n}\right) \left(\frac{n^2}{4} - \frac{n}{2} - 2\right) + \frac{1}{2} \left(\lg(n) - \frac{1}{n}\right) \left(\frac{3n^2}{4} - \frac{5n}{2} + 1\right) \quad (5)$$

$$= \frac{1}{2} \lg(n) \left(n^2 - \frac{7n}{2} - 1\right) - \frac{1}{2} \left(\frac{n^2}{4} - \frac{n}{2} - 2 + \frac{n}{2} - 1 - \frac{4}{n} + \frac{3n}{4} - \frac{5}{2} + \frac{1}{n}\right) \quad (6)$$

$$= \frac{1}{2} \lg(n) \left(n^2 - \frac{7n}{2} - 1\right) - \frac{1}{2} \left(\frac{n^2}{4} + \frac{3n}{4} - \frac{11}{2} - \frac{3}{n}\right) \quad (7)$$

$$= \frac{n^2}{2} \lg(n) - \frac{n^2}{8} - \frac{1}{2} \left(\lg(n) \left(\frac{7n}{2} + 1\right) + \frac{3n}{4} - \frac{11}{2} - \frac{3}{n}\right) \quad (8)$$

$$\leq \frac{n^2}{2} \lg(n) - \frac{n^2}{8} \quad \text{for } n \geq 2 \quad (9)$$