Recurrences – Suggested Solutions

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4.1-1 By induction on n, assuming $T(k) = O(\lg(k))$ for $k \leq \lceil n/2 \rceil$:

$$\begin{aligned} T(n) &\leq c\lg(\lceil n/2 \rceil) + 1 \\ &\leq c\lg(n/2 + 1) + 1 \\ &\leq c\lg(n+2) - c + 1 \end{aligned}$$

We want this expression to be $\leq c\lg n$. Let's see if it is possible and for which c:

$$\begin{aligned} & c\lg(n+2) - c + 1 \leq c\lg n \\ \Leftrightarrow\ & 1 \leq c(\lg n - \lg(n+2) + 1) \\ \Leftrightarrow\ & 1 \leq c(\lg\tfrac{n}{n+2} + 1) \end{aligned}$$

We need a c such that this holds for all $n \geq n_0$ from which we can apply the induction. We note that $\lg\frac{n}{n+2}$ is monotonic increasing, so if the expression holds for n_0 (yet to find), then it will hold for $n \geq n_0$. For $n = 2$ it does not work: $1 + \lg(2/4) = 0$; for $n = 3$: $1 + \lg(3/5) = 0.26\ldots$, we can find a c. For $c \geq \frac{1}{1+\lg(3/5)}$ we have:

$$T(n) \leq c\lg n$$

which complete the proof of the induction step. The base case is problematic as on page 64: check for $n = 2$ and $n = 3$ assuming $T(1) = 1$ and choose c large enough. Conclude by induction that $T(n) = O(\lg(n))$ for all $n \geq 2$.

4.1-2 By induction on n assuming $T(k) = \Omega(k\lg k)$ holds for $k \leq \lfloor n/2 \rfloor$:

$$\begin{aligned} T(n) &\geq 2c(\lfloor n/2 \rfloor)\lg(\lfloor n/2 \rfloor) + n \\ &\geq 2c(n/2 - 1)\lg(n/2 - 1) + n \\ &\geq c(n-2)\lg(n-2) - c(n-2) + n \\ &\geq c(n-2)\lg(n/2) - c(n-2) + n \text{ (for } n \geq 4) \\ &\geq c(n-2)\lg n - 2c(n-2) + n \\ &\geq cn\lg n - 2c\lg n - 2c(n-2) + n \\ &\geq cn\lg n + n(1-2c) + 4c - 2c\lg n \end{aligned}$$

We choose $n = 1/3$ and we examine $f(x) = 3 * (x(1-2c) + 4c - 2c\lg x) = x + 4 - 2\lg x$. $f'(x) = 1 - 2/x \geq 0 \Leftrightarrow x \geq 2$.

$$\begin{cases} f(2) = 2 + 4 - 2 > 0 \\ f'(x) \geq 0 \text{ for } x \geq 2 \\ f \text{ is continuous} \end{cases} \Rightarrow f(x) \geq 0 \text{ for } x \geq 2 \tag{1}$$

Thus for $n \geq 2$ and $c = 1/3$ we have $n(1 - 2c) + 4c - 2c\lg n \geq 0$ and therefore $T(n) \geq cn\lg n$. Base case $n = 1$: $T(1) \geq 0$. Conclude by induction that $T(n) = \Omega(n\lg n)$ for $n \geq 1$. Use theorem 3.1 to conclude $T(n) = \Theta(n\lg n)$.