Partial Knowledge in Membrane Systems. A Logical Approach

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Abstract. We propose a logic for specifying and proving properties of membrane systems. The main idea is to approach a membrane system by using the "point of view" of an external observer. Observers (as epistemic agents) accumulate their knowledge from the partial information they collect by observing subparts of the system and by applying logical reasoning to this information. We provide a formal framework to combine and interpret distributed knowledge in order to recover the complete knowledge about a membrane system. The proposed logic can be used to model biological situations where information concerning parts of the biological system is missing or incomplete.

1 Motivations

Abstracted as multi-agent system, a biological system reflects interactive, concurrent and distributed behaviors and, in general, the complex evolutions of biological systems. The success in dealing with this complexity depends on the mathematical model chosen as abstraction of the system.

Consider, for example, the *immune system* [2]. This is constituted by a network of cells, tissues and organs that work together to defend the body against attacks by foreign invaders - *microbes*, *germs*, *bacteria*, *viruses*, *parasites*, etc. The immune system's job is to keep them out or, failing that, to seek out and destroy them. The immune system functions due to an elaborate and dynamic communications network. Millions of cells, organized into sets and subsets, gather in clouds swarming around a hive and pass information back and forth.

Suppose now that we are interested in modelling the interaction of our body with a given virus. Excepting the immune system, our body contains also other subsystems, but we can decide that, for the given situation, all the other parts are meaningless. So we decide to ignore them. For instance, if we consider the case in which the virus is already present in our body, the first approximation of the biological reality will consider a main system (our body) in which are present two subsystems - the virus and the immune system. Going deeper, the first interaction between the two subsystems involves the *innate immune system*, which is just a subsystem of the immune system comprised of hereditary components that provide an immediate "first-line" of defense to continuously ward off pathogens. This subsystem is able to annihilate "well-known" viruses. If this is the real situation, then modelling only the innate immune system in relation with the virus is enough for comprehending the biological phenomenon. But if the virus is unknown, then we might need to go deeper with modelling and, in addition to the innate immune system, to model also *phagocytic cells*. These are cells that represents the "second-line" of defence for our body. They can analyze unknown entities, destroy viruses and learn the structures of the destroyed entities. In particular, the immune system is able to design special cells for fighting with peculiar types of viruses. Hence, on this level, the modelling have to be more specific representing also other subsystems of the immune system.

Depending on the complexity of the biological properties we want to consider, we can go as deep as necessary with representing the biological entities involved. More complex models provide more accurate information. Still, as the costs of modelling and simulation grows with complexity of the model, we have to find the right level of abstraction that gives, with acceptable costs, the information we are looking for. Observe that in biology, as in all the empirical sciences, *we cannot hope to reach the level of having complete information concerning a biological phenomenon.* Thus, no matter how complex is the model we choose, there exists always properties requiring a bigger complexity.

In other words, *we always work with partial (observed) knowledge* about biological systems and based on this incomplete information we model or simulate biological phenomena. In this paper we show how it is possible to manage incomplete information concerning membrane systems. The work done here can be seen as related to [9] where a formal observer has been introduced to investigate the formal behavior of a membrane system. However in [9] the observer was mainly used to extend the computing power of the observed device. In this paper, the observer is an epistemic agent able to compute knowledge and is used to analyze situations in which the knowledge about the observed system is partial, incomplete or missing.

2 Formal Language Preliminaries

Membrane systems are based on *formal language theory* and *multiset rewriting*. We now briefly recall the basic theoretical notions used in this paper. For more details the reader can consult standard books, such as [10] and the corresponding chapters of the handbook [18].

Given the set A we denote by $|A|$ its cardinality and by $\emptyset$ the empty set. We denote by $\mathbb{N}$ and by $\mathbb{R}$ the set of natural and real numbers, respectively.

As usual, an *alphabet* V is a finite set of symbols. By V^* we denote the set of all strings over V. By V^+ we denote the set of all strings over V excluding the empty string. The empty string is denoted by λ. The *length* of a string v is denoted by $|v|$. The concatenation of two strings $u, v \in V^*$ is written uv.

A *multiset* is a set where each element may have a multiplicity. Formally, a multiset over a set V is a map $M : V \rightarrow \mathbb{N}$, where $M(a)$ denotes the multiplicity of the symbol $a \in V$ in the multiset M.

For multisets M and M' over V, we say that M is *included in* M' if $M(a) \leq M'(a)$ for all $a \in V$. Every multiset includes the *empty multiset*, defined as M where $M(a) = 0$ for all $a \in V$.

The *sum* of multisets M and M' over V is written as the multiset $(M + M')$, defined by $(M + M')(a) = M(a) + M'(a)$ for all $a \in V$. The *difference* between M and M' is written as $(M - M')$ and defined by $(M - M')(a) = max\{0, M(a) - M'(a)\}$ for all $a \in V$. We also say that $(M + M')$ is obtained by *adding* M to M' (or viceversa) while $(M - M')$ is obtained by *removing* M' from M. For example, given the multisets $M = \{a, b, b, b\}$ and $M' = \{b, b\}$, we can say that M' is included in M, that $(M + M') = \{a, b, b, b, b, b\}$ and that $(M - M') = \{a, b\}$.

A multiset M can be expressed in the forms $(a, M(a))$ or $a^{M(a)}$, for all $a \in V$. If the set V is finite, e.g. $V = \{a_1, \ldots, a_n\}$, then the multiset M can be explicitly described as $\{(a_1, M(a_1)), (a_2, M(a_2)), \ldots, (a_n, M(a_n))\}$. The *support* of a multiset M is defined as the set $supp(M) = \{a \in V \mid M(a) > 0\}$. A multiset is empty (hence finite) when its support is empty (also finite).

A compact notation can be used for finite multisets: if $M = \{(a_1, M(a_1)), (a_2, M(a_2)), \ldots, (a_n, M(a_n))\}$ is a multiset of finite support, then the string $w = a_1^{M(a_1)} a_2^{M(a_2)} \ldots a_n^{M(a_n)}$ (and all its permutations) precisely identify the symbols in M and their multiplicities. Hence, given a string $w \in V^*$, we can say that it identifies a finite multiset over V, written as $M(w)$, where $M(w) = \{a \in V \mid (a, |w|_a)\}$. For instance, the string *bab* represents the multiset $M(w) = \{(a, 1), (b, 2)\}$, that is the multiset $\{a, b, b\}$. The empty multiset is represented by the empty string λ.

3 Membrane Systems with Symbol-Objects

We recall the basic notions of membrane systems (also called P systems) with symbol-objects. The reader can find further details in the monograph [3]. An updated bibliography of the field can be found at the P systems web-page [1].

Definition 1 (Membrane system with symbol-objects). *Given a finite set of objects O and an infinite set of labels Lab, we consider the following family of constructs*

$$\mathbb{P} = \{(\mu, w_{j_1}, w_{j_2}, \cdots, w_{j_m}, R_{j_1}, R_{j_2}, \cdots, R_{j_m}) \mid j_i \in Lab,\ for\ i = 1..m\}.$$

where

- *μ is a* membrane structure *consisting of m membranes arranged in an hierarchical structure enclosed in a main membrane, called* skin membrane*. The skin membrane separates the system from the surrounding* environment*; the membranes (and hence the regions that they delimit/enclose) are injectively labeled over $Lab_{\Pi} = \{j_1, j_2, \cdots, j_m\} \subset Lab$; we* convey to label by j_1 the skin membrane*.*

- $w_{j_1}, w_{j_2}, \cdots, w_{j_m}$ *are strings that represents multisets over* O *associated with regions* $j_1, j_2, \cdots, j_m$*, respectively.*
- $R_{j_1}, R_{j_2}, \cdots, R_{j_m}$ *are finite sets of* evolution rules *over* O*, associated to regions* $j_1, j_2, \cdots, j_m$*, respectively. An evolution rule is of the form* $u \rightarrow v$*, where* u *is a string over* O *and* v *is a string over* $\{a_{here}, a_{out} \mid a \in O\} \cup \{a_{in_I} \mid a \in O, I \subseteq Lab\}$*. The symbols* here, out, in_I with $I \subseteq Lab$ *are called* target indications. *To simplify the notation the target indication* here *is omitted.*

An element $\Pi = (\mu, w_{j_1}, w_{j_2}, \cdots, w_{j_m}, R_{j_1}, R_{j_2}, \cdots, R_{j_m}) \in \mathbb{P}$ is called *membrane system with symbol-objects*, of degree m. We denote by 0 the membrane system of degree 0. We call *atomic membrane system* a system a system of degree 1 having either the set of rules empty or the multiset of objects empty; if its unique membrane (which is also the skin membrane) is labelled by i and $R_i = \emptyset$ while its multiset is $w_i \in O^*$, then we denote it by $[w_i]_i$; if $w_i = \lambda$ and $R_i \neq \emptyset$ then we denote it by $[R_i]_i$; if $R_i = \emptyset$ and $w_i = \lambda$ we convey to denote it by $[0]_i$.

Given a membrane system Π, an *evolution of* Π is a *sequence of membrane systems* $\langle \Pi_0, \Pi_1, \Pi_2, \cdots \rangle$ where $\Pi_0 = \Pi$ and, for $i \geq 0$, each Π_{i+1} is obtained by applying once one of the associated evolution rules in one of the regions of Π_i. Rule and region are chosen in a non-deterministic manner. The remainder of the system Π_i (objects not involved in the application of the rule, set of rules, membrane structure, labelling of the membranes) is left unchanged in Π_{i+1}.

The passage from Π_i to Π_{i+1} using the rule r in region j of Π_i is called *transition* and is denoted by $\Pi_i \xrightarrow{r_j} \Pi_{i+1}$. [1]

The *application of an evolution rule* $r : u \rightarrow v \in R_j$ in the region $j \in Lab_\Pi$ means to remove the multiset of objects identified by u from region j, and to add the objects specified by the multiset v, in the regions specified by the target indications associated to each object in v. In particular, if v contains an object a with target indication *here*, then the object a will be placed in the region j where the evolution rule has been applied. If v contains an object a with target indication *out*, then the object a will be moved to the region immediately outside the region j (this can be the environment if the region where the rule has been applied is the *skin* membrane). If v contains an object a with target indication in_I, with $I \subseteq Lab$, then the object a is moved from the region j and placed in a non-deterministic way into a region $i \in I$ (this can be done only if such region $i \in I$ is immediately inside region j; otherwise the evolution rule $u \rightarrow v$ cannot be applied).

We call *contents of membrane* j *in* Π, the multiset of objects and the membranes (together with their contents) contained in region j of Π.

[1] The reader familiar with membrane systems can notice that we use a sequential semantics: at each step only a unique rule is executed once. Actually the logic proposed in this paper is very general and can be extended easily to other semantics, e.g., the maximal parallel one.

Definition 2 (Membrane composition).

Let $\Pi = (\mu, w_{j_1}, w_{j_2}, \cdots, w_{j_m}, R_{j_1}, R_{j_2}, \cdots, R_{j_m})$ *be a membrane system and* $i \in Lab - Lab_\Pi$.

We denote by $[\Pi]_i$ *the membrane system*

$$\Pi' = (\mu', w_{k_1}, w_{k_2}, \cdots, w_{k_{m+1}}, R_{k_1}, R_{k_2}, \cdots, R_{k_{m+1}})$$

such that

- μ' *is* μ *enclosed into an external membrane labelled by* i*; the labelling of the membranes of* μ *is preserved in* μ'*;*
- $k_1 = i$ *and* $k_s = j_{s-1}$ *for* $s = 2..m+1$*; consequently* $w_{k_s} = w_{j_{s-1}}$ *and* $R_{k_s} = R_{j_{s-1}}$*;*
- $w_{k_1} = \lambda$*,* $R_{k_1} = \emptyset$*.*

Example 1. Consider the membrane system Π defined by

$$\mu = [\ [\ ]_2\ ]_1;$$
$$R_1 = \{a \to b\};$$
$$R_2 = \{b \to c\};$$
$$w_1 = b;$$
$$w_2 = a.$$

Then $[\Pi]_3$ is the system Π' defined by

$$\mu' = [\ [\ [\ ]_2\ ]_1\ ]_3;$$
$$R'_1 = R_1 = \{a \to b\};$$
$$R'_2 = R_2 = \{b \to c\};$$
$$R'_3 = \emptyset;$$
$$w'_1 = w_1 = b;$$
$$w'_2 = w_2 = a;$$
$$w'_3 = \lambda.$$

Definition 3 (Parallel composition).

Let $\Pi = (\mu, u_{j_1}, u_{j_2}, \cdots, u_{j_m}, R_{j_1}, R_{j_2}, \cdots, R_{j_m})$ *and* $\Pi' = (\mu', v_{k_1}, v_{k_2}, \cdots, v_{k_n}, R_{k_1}, R_{k_2}, \cdots, R_{k_n})$ *be two membrane systems such that* $j_1 = k_1$ *and* $Lab_\Pi \cap Lab_{\Pi'} = \{j_1\}$*. We call parallel composition of the two systems, denoted by* $\Pi|\Pi'$*, the membrane system*

$$\Pi'' = (\mu'', w_{l_1}, w_{l_2}, \cdots, w_{l_{m+n-1}}, R_{l_1}, R_{l_2}, \cdots, R_{l_{m+n-1}})$$

defined by:

- μ'' *is obtained by enclosing into a common external membrane the contents of the skin membranes of* μ *and* μ'*;*

- *in μ'' the labelling of the membranes in μ and in μ' is preserved; consequently the skin membrane of Π'' is labelled by $l_1 = j_1 = k_1$;*
- $w_{l_1} = u_{j_1} v_{k_1}, R_{l_1} = R_{j_1} \cup R_{k_1}$. [2]

The intuition behind the parallel composition operator is that it can be used to divide an entire membrane system in subsystems, where each subsystem can be recognized/understood by a certain external observer.

Example 2. Consider the membrane systems

Π :	Π' :
$\mu = [\,[\,]_2\,[\,]_3\,]_1$	$\mu' = [\,[\,[\,]_5\,]_4\,]_1$
$w_1 = ab$	$w_1 = ee$
$w_2 = cd$	$w_4 = ccd$
$w_3 = aa$	$w_5 = a$
$R_1 = \{a \to b,\ a \to c\}$	$R_1 = \{a \to b, a \to d\}$
$R_2 = \{cd \to a\}$	$R_4 = \{d \to c\}$
$R_3 = \{a \to b, a \to d\}$	$R_5 = \{a \to b\}$

Then $\Pi|\Pi'$ is the system Π'' defined as

$$\mu'' = [\,[\,]_2\,[\,]_3\,[\,[\,]_5\,]_4\,]_1$$
$$w_1 = eeab$$
$$w_2 = cd$$
$$w_3 = aa$$
$$w_4 = ccd$$
$$w_5 = a$$
$$R_1 = \{a \to b, a \to c, a \to d\}$$
$$R_2 = \{cd \to a\}$$
$$R_3 = \{a \to b, a \to d\}$$
$$R_4 = \{d \to c\}$$
$$R_5 = \{a \to b\}.$$

Let denote by $\mathbb{P}_i$ the class of membrane systems having the skin membrane labelled by i. Then it is easy to see that the following theorem holds.

Theorem 1. $(\mathbb{P}_i, |, [0]_i)$ *is an Abelian monoid.*

Also the following theorem can be easily proved.

Theorem 2. *Any membrane system can be composed, by iterating parallel and membrane composition, starting from atomic membrane systems.*

[2] The definition is correct as $Lab_\Pi \cap Lab_{\Pi'} = \{j_1\}$. Notice that, since the labelling of the membranes is preserved, we have that for $s \neq 1$ R_{l_s} and $w_{l_s} = u_{k_s}$ ($w_{l_s} = v_{k_s}$) are preserved as in the original system Π (Π', respectively)

Example 3. Consider the membrane system Π presented in Example 2.

The system Π can be obtained as

$$[\ [cd]_2|[R_2]_2\ ]_1 \mid [\ [aa]_3|[R_3]_3\ ]_1 \mid [ab]_1 \mid [R_1]_1$$

with R_1, R_2 and R_3 as in Π. Clearly $[cd]_2$, $[aa]_3$, $[ab]_1$, $[R_1]_1, [R_2]_2, [R_3]_3$ are atomic membrane systems.

Partial Information in Membrane Systems

We want to propose a formal way of playing with *partial information* about a (membrane) system in order to decide some global properties. The idea is to formally describe *open systems*. An open system for an observer is a system formed by a known subsystem and an unknown (opened) part about which the observer does not know anything. So if the observer knows a subsystem S_1 of a bigger system $S_1|S_2$, then the observer considers as entire system, any structure of type $S_1|S_3$, for any possible system S_3. Hence, the properties that the observer knows about the entire system are the properties that systems "like" $S_1|S_2$, $S_1|S_3$, etc. have in common.

Consider again the example, presented in the Introduction, where a virus attacks our body. We have decided to model a relevant part of immune system, say I, in relation with the virus v. Hence the model of a body that has been penetrated by a virus is $body = I|v|S$, where S denotes the rest of the body (we have not considered to model the rest of the body in details since the system I is enough for comprehending the interaction with the virus). Suppose now that the properties we try to specify do not concern only the subsystem $I|v$ (the one we have considered) but the whole body $I|v|S$.

Can we sustain that each property of the system $I|v$ can be stated about the whole body $I|v|S$?

For correctly answering to this question, we propose a logic to play with partial information. Consider a complex biological system about which we have only partial information. This information is collected by some observers placed in different points of the system. Each observer analyzes a subsystem. Our logic develops the framework needed to combine the knowledge of these observers such that is possible to derive interesting properties about the whole system, even without having complete information about it. Playing with observers might cost less then fully investigating the system and it might provide enough information for deciding on the properties we are interested in. All depends on how we place the observers and how we combine their knowledge in deriving complex properties.

Formally, we propose a logic developed in dynamic-epistemic paradigm [11] and enriched with operators from spatial logics [5, 4, 7, 8]. We call it *dynamic epistemic spatial logic*. The syntax allows to express open systems and the knowledge of observers. By combining the knowledge of different observers we can specify and verify complex properties about the whole system without having complete knowledge about it.

In related papers [15, 16, 14] it has been proposed Hilbert-style axiomatic systems for different such logics, and it has been proved that they are decidable against a semantics based on process algebra, even in the cases for which the classical spatial logics have been proved to be undecidable [6].

4 Playing with Partial Information

In this section we will show how, playing with partial information about a system, we can derive properties of the whole system. For this we reconsider a classical example used in epistemic reasoning [11] adapted for a biologically inspired situation.

Consider a biological system S composed by four disjoint subsystems $S = S_1|S_2|S_3|S_4$. In Figure 1 there are four cells S_1, S_2, S_3 and S_4. Each cell contains a vacuole that can be either normal, having an oval shape as in S_3, S_4, or abnormal having a non-circular shape as the vacuoles of S_1 and S_2. Suppose, in addition, that the system is analyzed by four observers, each observer having access to only a subpart of S. Thus observer O_1 can see the subsystem $S_2|S_3|S_4$, O_2 the subsystem $S_3|S_4|S_1$, O_3 can see the subsystem $S_4|S_1|S_2$ and observer O_4 sees $S_1|S_2|S_3$. Each observer has a display used for making public announcements.

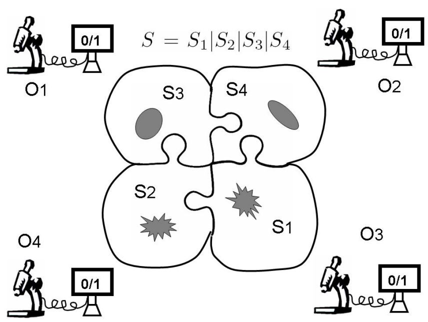

Fig.1: The system S

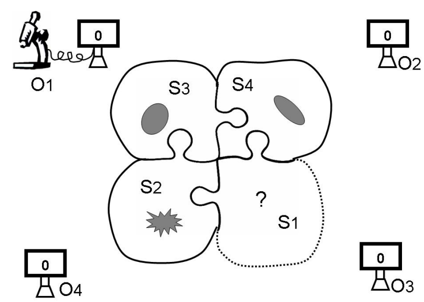

Fig.2: The perspective of O_1

The observers know that the system S contains abnormal vacuoles and each observer tries to compute the exact number of them and their positions in the system. In doing this the observers do not communicate but only witness the public announcements. Each observer displays 0 until it knows the exact number and positions of abnormal vacuoles, moment in which it switches to 1. In addition, the observers are synchronized by a clock that counts each step of computation. Hence, after each "tic" the observer has to evaluate its knowledge and to decide if its display remains on 0 or switches to 1. Thus each observer computes information about the whole system by using the partial information it possesses and by evaluating the knowledge of the other observers (by reading their displays). If an observer is able to decide the correct number of abnormal vacuoles and their exact positions in the system, then it succeeded to do this with a lower cost then the cost needed for fully investigating the system. Hereafter we show that such a deduction is possible.

Consider that the real state of the system is the one in Figure 1. And suppose that we can control only the observer O_1. As O_1 sees the subsystem $S_2|S_3|S_4$, it sees an abnormal vacuole in subsystem S_2 and normal vacuoles in S_3 and S_4; in Figure 2 it is represented the perspective of O_1. But it does not know if the system S_1 has a normal or an abnormal vacuole. For O_1 both situations are equally possible. Hence, after the first round of computation the display of O_1 remains to 0. As it concerns observer O_2, it sees an abnormal vacuole, in S_1, but it doesn't know what is in S_2, thus, after the first round, it will show 0 too.

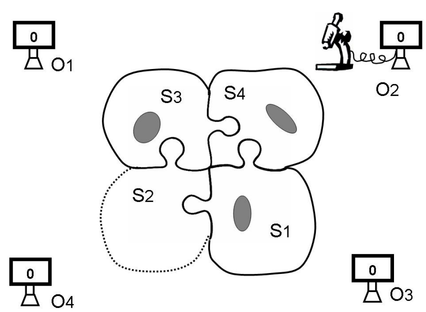

Fig.3: A hypothetical perspective of O_2

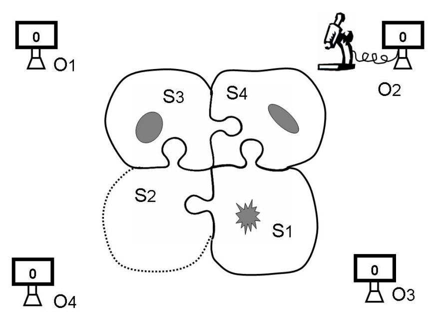

Fig.4: The real perspective of O_2

It starts the second round of computation. We come back to our observer, O_1. The observer has seen that, after the first round, the observer O_2 has not succeed to understand the situation (as O_2 shows 0 on its display). If the system S_1 would contain a normal vacuole then in the first round O_2 would have seen only normal vacuoles, as in Figure 3. O_2 also knows that there is at least one abnormal vacuole. Hence, if this was the case, O_2 had enough information to decide, in the first round, that the only abnormal vacuole of the system is in S_2. Consequently 1 had to appear on its display. But this was not the case (O_1 can see that by looking to the display of O_2). This means that what O_2 observed was the situation presented in Figure 4. Therefore it is possible to decide that the real situation of the system is the one with an abnormal vacuole in S_1. Thus using *only* O_1 *it is possible to compute the real configuration of the system* and then O_1 will display 1. The example works similarly in more complex situations.

Observe the advantages of this analysis: using only the partial information available to O_1 about the system S and judging the behavior of the other observers, we were able to compute the real configuration of the system. The observers do not exchange information about S, but only about their level of understanding (their observations of) S. The rest can be computed. If each subsystem is very complex, and usually this is the case in biology, then the complete information about the system can be larger then an observer can store or manipulate.

Note also that the observers do not need a central unit for organizing their information. Each observer organizes its own information and makes public announcements about its level of knowledge. They work simultaneously in a distributed network and only playing with their partial information about S and with the information about the state of the network are able to derive overall properties of the system.

The approach fits well with the real situation of biological systems. We work always with partial information which are collected by some observers as results of "measuring" different aspects of a biological phenomena. Sometimes these different "faces" of the same phenomena cannot be integrated in the same mathematical model, or seeking for a property might involve evaluation of different models. For such situations, a formalized way for automatically reasoning, as in the previous example, might help. Hereafter we introduce a logic designed for this purpose.

5 A Logic of Partial Information

As pointed in the previous section, the role of observers in understanding and manipulating biological information is significant. We present a logic of observers, called dynamic epistemic spatial logic [15, 17, 16, 14], developed for specifying and model-checking properties of multi-agent systems. It can be successfully applied for analyzing membrane systems. Our logic proposes a formal way of combining and analyzing the information provided by different observers about the same biological phenomena.

Our logic can be related with spatial logics [4, 5, 7, 8]. For a detailed presentation of it and for a Hilbert-style axiomatization the reader is referred to [15, 16, 14].

5.1 The Syntax of $\mathcal{L}_{Obs}$

Suppose that we have a class *Obs* of observers ranged over by A, B, C. We enrich the language of propositional logic with knowledge operators indexed by observers K_A. A statement like $K_A\phi$ is read *"observer A knows ϕ"*. Then we can compose more complex epistemic statements. Thus *"observer A_1 knows that observer A_2 knows ϕ"* is formalized by $K_{A_1}K_{A_2}\phi$. A formula like $K_A\phi \wedge K_A(\phi \rightarrow \psi) \rightarrow K_A\psi$ is interpreted as *"if observer A knows ϕ and $\phi \rightarrow \psi$ then the observer knows ψ"*.

In addition to these operators we add some spatial operators meant to describe the spatial distribution of the subsystems. Anticipating the semantics, we present the intuition behind these operators.

Formula 0 is meant to characterize the trivial membrane system 0 that might be ignored in a complex situation[3].

[3] Some syntaxes of classical logic use 0 for denoting *false*. This is not the case here. We use $\perp$ to denote *false*.

Inspired by spatial logics [5, 7, 8], we introduce the parallel operator $\phi \parallel \psi$ meant to express the situation in which our system can be decomposed in two (parallel) subsystems, one satisfying ϕ and the other one satisfying ψ.

$\top$ will be satisfied by any system, hence it expresses consistency, "*true*". The role of this element of syntax is essential in expressing open systems. As $\top$ is a property that characterizes any system, $\phi \parallel \top$ characterizes any system that has a subsystem satisfying ϕ and the rest of the system is, possibly, unknown.

By negation, $\bot$ will be used to express the inconsistency.

We also design operators to express membranes. Thus $[\![\phi]\!]_i$ is a property that characterizes a membrane system $[\Pi]_i$ where Π is a membrane system that has the property ϕ. Similarly we introduce formulas $[\![w_i]\!]_i$ and $[\![R_i]\!]_i$ that characterize the atomic membrane systems $[w_i]_i$ and $[R_i]_i$ respectively.

As we propose a logic for the studying of membrane systems together with their evolutions, we allow some modal operators indexed by the transitions of the systems to express the evolutions of a membrane system. Thus $\langle r_i \rangle \phi$ is an operator meant to specify the system Π able to perform a transition r_i, i.e. $\Pi \xrightarrow{r_i} \Pi'$, and Π' satisfies ϕ. These operators are inspired by dynamic logics [12] and are basic operators in Hennessy-Milner logic [13].

Formally, the language of dynamic epistemic spatial logic $\mathcal{L}_{Obs}$ is defined as follows:

Definition 4 (The language). *Let Obs be the set of observers, O an alphabet and Lab a set of labels. We define the language of dynamic epistemic spatial logic, by the following grammar:*

$$\phi := \ 0 \mid \ \top \ \mid \ \neg\phi \ \mid \ \phi \wedge \phi \ \mid \ [\![w_i]\!]_i \ \mid [\![R_i]\!]_i \mid \ [\![\phi]\!]_i \ \mid \ \phi \parallel \phi \ \mid \ \langle r_i \rangle \phi \ \mid \ K_A \phi$$

where $A \in Obs$, $w \in O^*$, $i \in Lab$ and r_i is a rule of the set R_i.

Definition 5 (Derived operators). *In addition we introduce some derived operators, widely used in dynamic-epistemic logics:*

1. $\bot \stackrel{def}{=} \neg\top$
2. $\phi \vee \psi \stackrel{def}{=} \neg((\neg\phi) \wedge (\neg\psi))$
3. $\phi \rightarrow \psi \stackrel{def}{=} (\neg\phi) \vee \psi$
4. $[r_i]\phi \stackrel{def}{=} \neg(\langle r_i \rangle (\neg\phi))$
5. $1 \stackrel{def}{=} \neg((\neg 0) \ \parallel \ (\neg 0))$
6. $\widetilde{K}_A \phi \stackrel{def}{=} \neg K_A \neg\phi$

The dynamic modality $[r_i]\phi$, the dual operator of $\langle r_i \rangle \phi$, captures the weakest precondition of a transition r_i of a membrane system w.r.t. a given post-specification ϕ. We have used the square brackets to denote it, as this notation is classical in dynamic logics (inspired by the box operator of modal logic). It shouldn't be confused with the same brackets use on membrane systems for denoting membrane composition.

The formula 1 is meant to describe the situation in which the system cannot be decomposed into two non-trivial subsystems.

5.2 The Semantics of $\mathcal{L}_{Obs}$

In this subsection we introduce a semantics for the presented logic. It will be defined underpinning on a *satisfiability relation* $\Pi \models \phi$, that establishes the condition under which we can affirm that the membrane system Π has (satisfies) the property ϕ.

As introduced earlier, each observer sees a membrane system in $\mathbb{P}$. This membrane system is the "structure" that the observer can recognize in any more complex system. Hence, for introducing the semantics, we have to devise an *interpretation function* $int : Obs \longrightarrow \mathbb{P}$ that associates to each observer $A \in Obs$ a membrane system $int(A) = \Pi$ that represents the system that the observer is able to "recognize". The intuition is to define the knowledge of the observer A as the common properties of all systems where A is *active*, i.e., systems that contains Π as subsystem.

Definition 6 (Models and satisfaction). *Given a class Obs of observers and an interpretation function* $int : Obs \longrightarrow \mathbb{P}$ *we introduce the satisfaction relation by:*

$\Pi \models \top$ *always*

$\Pi \models \neg\phi$ *iff* $\Pi \not\models \phi$

$\Pi \models \phi \wedge \psi$ *iff* $\Pi \models \phi$ *and* $\Pi \models \psi$

$\Pi \models \phi \parallel \psi$ *iff* $\Pi = \Pi_1|\Pi_2$ *and* $\Pi_1 \models \phi$, $\Pi_2 \models \psi$

$\Pi \models 0$ *iff* $\Pi = 0$

$\Pi \models [\![w_i]\!]_i$ *iff* $\Pi = [w_i]_i$

$\Pi \models [\![R_i]\!]_i$ *iff* $\Pi = [R_i]_i$

$\Pi \models [\![\phi]\!]_i$ *iff* $\Pi = [\Pi']_i$ *and* $\Pi' \models \phi$

$\Pi \models \langle r_i \rangle \phi$ *iff there exists a transition* $\Pi \xrightarrow{r_i} \Pi'$ *and* $\Pi' \models \phi$

$\Pi \models K_A\phi$ *with* $int(A) = \Pi'$ *iff* $\Pi = \Pi'|\Pi''$ *and* $\forall \Pi'|\Pi''' \in \mathbb{P}$ *we have* $\Pi'|\Pi''' \models \phi$

Then the semantics of the derived operators can be obtained.

$\Pi \models [r_i]\phi$ iff for any Π' such that $\Pi \xrightarrow{r_i} \Pi'$ (if any), $\Pi' \models \phi$

$\Pi \models 1$ iff there are no systems Π', Π'' with $\Pi = \Pi'|\Pi''$ and $\Pi' \neq 0 \neq \Pi''$

$\Pi \models \widetilde{K}_A\phi$ for $int(A) = \Pi'$ iff either $\Pi \neq \Pi'|\Pi''$ for any Π'', or it exists $\Pi'|\Pi'''$ such that $\Pi'|\Pi''' \models \phi$

5.3 Expressivity

Open systems: We can exploit the use of $\top$ to express properties of *open membrane systems*. By an open membrane system we mean a system for which we know only a subpart and we accept any upper-system of the known part as possible configuration for the overall situation. For example if our system is $\Pi = \Pi_1|\Pi_2$ and an observer knows only Π_1, then for the observer any system of type $\Pi_1|\Pi_3$, for any $\Pi_3 \in \mathbb{P}$, is a possible system Π. Hence what is outside Π_1 is "open information" for our observer. Reconsidering the example in section

4, for A_1, in the initial state, Π was an open system because Π_1 has (for A_1) either a normal, or an abnormal vacuole.

If we want to express that a system Π is an open system containing a known subsystem Π_1 then we can express this by allowing an observer $A_1 \in Obs$ to see only Π_1, i.e. $int(A_1) = \Pi_1$. Then $\Pi \models K_{A_1}\top$ means that the system Π is an upper system of Π_1. Indeed, by our semantics, this means that $\Pi = \Pi_1|\Pi_2$ and for any $\Pi_3 \in \mathbb{P}$ we have $\Pi_1|\Pi_3 \models \top$. But the last condition is trivially true, hence the semantics is equivalent to $\Pi = \Pi_1|\Pi_2$, where Π_2 can be any system. Due to this, we can use $K_{A_1}\top$ to say "in this system Π_1 is a subsystem".

We can be more specific and express that any upper system of Π_1 has the property ϕ. We can do this by taking an upper system of Π_1, say $\Pi = \Pi_1|\Pi_2$, and stating that $\Pi \models K_{A_1}\phi$, where $int(A_1) = \Pi_1$. This is equivalent with saying that for any $\Pi_3 \in \mathbb{P}$ we have $\Pi_3|\Pi_1 \models \phi$.

If we can characterize a membrane system up to identity, we can express that a system Π is an open system containing a known subsystem characterized by ϕ also without using the epistemic operator, by $\Pi \models \phi \parallel \top$. Indeed, w.r.t. our semantics this means that $\Pi = \Pi_1|\Pi_2$ and $\Pi_1 \models \phi$, $\Pi_2 \models \top$. As ϕ satisfies the known system and $\top$ can be stated about any system $\Pi_3 \in \mathbb{P}$ we obtain that any system of type $\Pi_1|\Pi_3$, for any $\Pi_3 \in \mathbb{P}$ satisfies $\phi \parallel \top$.

Characteristic formulas: Using our logic we can define formulas that will fully identify a membrane system. Recall Theorem 2 stating that each membrane system can be decomposed, by using parallel and membrane composition, in atomic membrane systems. We show further how, by induction in top of atomical membrane systems, we can define characteristic formulas for any membrane system.

A characteristic formula of a membrane system Π have to be a formula of our logic ϕ_Π such that

- $\Pi \models \phi_\Pi$
- if $\Pi' \models \phi_\Pi$ then $\Pi' = \Pi$

We define such formulas inductively on structure of Π.

1. $\phi_0 \stackrel{def}{=} 0$
2. $\phi_{[w]_i} \stackrel{def}{=} [\![w]\!]_i$

$2'$. $\phi_{[R]_i} \stackrel{def}{=} [\![R]\!]_i$

3. $\phi_{\Pi|\Pi'} \stackrel{def}{=} \phi_\Pi \parallel \phi_{\Pi'}$
4. $\phi_{[\Pi]_i} \stackrel{def}{=} [\![\phi_\Pi]\!]_i$

Theorem 3. *Giving an arbitrary membrane system Π, the formula ϕ_Π is a characteristic formula for Π.*

The fact that we can define characteristic formulas for membrane systems open the possibility to project any semantical problem in syntax. Thus, if we want to verify that the system Π has a property ψ, i.e. $\Pi \models \psi$, we can project this problem in syntax where it is equivalent with $\vdash \phi_\Pi \to \psi$, where we denoted by ϕ_Π the characteristic formula of Π as before. Now the problem $\Pi \models \psi$ is equivalent with proving $\phi_\Pi \to \psi$ with the axioms of our logic.

Similarly, we can express the fact that between Π and Π' there exists a transition $\Pi \xrightarrow{r_i} \Pi'$ by stating $\vdash \phi_\Pi \rightarrow \langle r_i \rangle \phi_{\Pi'}$. Now $\vdash \phi_\Pi \rightarrow \langle r_i \rangle \phi_{\Pi'}$ can be proved from the axioms iff the transition $\Pi \xrightarrow{r_i} \Pi'$ exists. On this direction we can also imagine more complex situations. Consider, for example, that we have the system Π and we want to know if, after doing the transitions labelled by r_i then s_j, the t_s it will reach a state (a membrane system) that will have a subpart satisfying ψ. This can be syntactically said by $\vdash \phi_\Pi \rightarrow \langle r_i \rangle \langle s_j \rangle \langle t_s \rangle (\psi \parallel \top)$. Indeed $\psi \parallel \top$ describes a system having a subsystem that satisfies ψ. Then the dynamic operators prefixing it, $\langle r_i \rangle \langle s_j \rangle \langle t_s \rangle (\psi \parallel \top)$, means that the system will reach the state satisfying $\psi \parallel \top$ only after it performs the transitions labelled by r_i, s_j, t_s in this order.

Validity: The presented syntax allows to express the validity of a property in a class of membrane systems having the same external membrane i, i.e. the property is satisfied by any of these systems. We can do this by using a "*blind observer*", i.e. an observer $A' \in Obs$ that sees only the trivial system embedded in i, $int(A') = [0]_i$.

Indeed, the epistemic operator $K_{A'}$ has the following semantics.

$\Pi \models K_{A'}\phi$ iff for any $\Pi'' \in \mathbb{P}_i$ we have $\Pi'' \models \phi$.

This is so because, if a system Π has the property $K_{A'}\phi$ then ϕ is satisfied by any system $\Pi' \in \mathbb{P}$ that can be decomposed in $\Pi' = [0]_i|\Pi''$, i.e. Π' must have the skin membrane i, hence $\Pi' \in \mathbb{P}_i$. But Π' has the property $\Pi'|[0]_i = \Pi'$, as $[0]_i$ is the null element of the monoid $(\mathbb{P}_i, |)$. Hence ϕ is satisfied by any system with the skin i, i.e. it is a valid property over $\mathbb{P}_i$. Thus we can encode, in syntax, the validity of a property.

Consequently, $K_{A'}\top$ is a validity, as $[0]_i$ is a subsystem of any system in $\mathbb{P}_i$, $\Pi = \Pi|[0]_i$.

Satisfiability: Also the satisfiability of a property can be encoded in the syntax. We say that a property is satisfiable if it exists at least one membrane system having this property. For this purpose we use the dual of the knowledge operator for the blind observer $\widetilde{K}_{A'}$ (as before we assume that $int(A') = [0]_i$).

$\Pi \models \widetilde{K}_{A'}\phi$ iff it exists a membrane system $\Pi'' \in \mathbb{P}_i$ such that $\Pi'' \models \phi$

Indeed, if a system Π satisfies $\widetilde{K}_{A'}\phi$ then either $\Pi \neq \Pi'|[0]_i$ (this is not the case as always $\Pi = \Pi|[0]_i$) or it exists Π'' such that $\Pi''|[0]_i \models \phi$. But $\Pi''|[0]_i = \Pi''$, hence it exists a system $\Pi'' \in \mathbb{P}_i$ that satisfies ϕ and vice versa. Thus $\widetilde{K}_{A'}\phi$ provides a way to encode, in syntax, the satisfiability of a property.

5.4 (Some) Axioms, Rules and Theorems

In [15, 17, 16, 14] it has been introduced a Hilbert-style axiomatic system for dynamic epistemic spatial logic. We present further some interesting axioms and theorems that can offer an idea about what can be specified and proved using our logic.

Axiom A 1 $\vdash [\![\phi]\!]_i \parallel [\![0]\!]_i \leftrightarrow [\![\phi]\!]_i$

The previous axioms states that an empty membrane system contained in membrane i do not come with extra properties if it is considered as a subsystem of a system having the skin i. Hence, such a subsystem is "transparent".

Axiom A 2 $\vdash \phi \parallel \psi \rightarrow \psi \parallel \phi$

Axiom A 3 $\vdash (\phi \parallel \psi) \parallel \rho \rightarrow \phi \parallel (\psi \parallel \rho)$

These entail that $\parallel$ organizes an abelian monoid structure.

Rule R 1 *If* $\vdash \phi \rightarrow \psi$ *then* $\vdash \phi \parallel \rho \rightarrow \psi \parallel \rho$.

This rule establish the monotonicity of parallel composition.

Axiom A 4 $\vdash [r_i](\phi \rightarrow \psi) \rightarrow ([r_i]\phi \rightarrow [r_i]\psi)$.

This axiom is the (K) axiom well-known in modal and dynamic logics which, together with the next rule of necessity shows that, indeed, our operator is an authentic modal operator.

Rule R 2 *If* $\vdash \phi$ *then* $\vdash [r_i]\phi$.

Axiom A 5 $\vdash (\langle r_i \rangle \phi) \parallel \psi \rightarrow \langle r_i \rangle (\phi \parallel \psi)$.

If a subsystem Π_1 of a system $\Pi = \Pi_1 | \Pi_2$ can do a transition r_i and further it satisfies ϕ while its counterpart Π_2 satisfies ψ, then the system Π can be described as able to perform a transition r_i thus passing to a system satisfying $\phi \parallel \psi$.

Axiom A 6 $\vdash K_A\phi \wedge K_A(\phi \rightarrow \psi) \rightarrow K_A\psi$

This axiom A6 is the classical (K)-axiom stating that our epistemic operator is a normal one. It states that if an observer A knows ϕ and that $\phi \rightarrow \psi$ then it knows ψ. It is an usual axiom of knowledge [11].

Axiom A 7 $\vdash K_A\phi \rightarrow \phi$

Also this axiom is classic in modal and epistemic logics - the axiom (T) - necessity axiom. It states that the knowledge of any observer must be true, i.e. an observer cannot know something that is not true.

Axiom A 8 $\vdash K_A\phi \rightarrow K_AK_A\phi$.

Also axiom A8 is well known in epistemic logics. It states that our epistemic agents (observers) have *the positive introspection property*, i.e. if an observer A knows something then it (i.e., the observer) knows that it knows that thing.

Axiom A 9 $\vdash K_A\top \rightarrow (\neg K_A\phi \rightarrow K_A\neg K_A\phi)$

Axiom A9 states a variant of *negative introspection*, saying that if an observer A is active (the system that the observer knows is a subsystem of the whole system) and if the observer does not know ϕ, then the observer knows that does not know ϕ. Negative introspection is also present in classic epistemic logics.

Rule R 3 *If* $\vdash \phi$ *then* $\vdash K_A\top \to K_A\phi$.

Rule R3 states that any active observer knows all the tautologies. Also in this case we deal with a well known epistemic rule, widely spread in epistemic logics. But our rule works under the assumption that the observer is active.

In [15, 16, 14] we present a complete axiomatic system and we prove many theorems in it. Hereafter we will sketch some soundness proofs for the previous axioms to clarify the intuitions that motivates the choice of them. Similarly all the axioms can be proved to be sound.

Theorem 4 (Soundness of axiom A5). $\models (\langle r_i\rangle\phi) \parallel \psi \to \langle r_i\rangle(\phi \parallel \psi)$

Proof. If $\Pi \models (\langle r_i\rangle\phi) \parallel \psi$, then $\Pi = \Pi_1|\Pi_2$, $\Pi_1 \models \langle r_i\rangle\phi$ and $\Pi_2 \models \psi$. So $\exists\Pi_1 \xrightarrow{r_i} \Pi_1'$ and $\Pi_1' \models \phi$. So $\exists\Pi = \Pi_1|\Pi_2 \xrightarrow{r_i} \Pi' = \Pi_1'|\Pi_2$ and $\Pi' \models \phi \parallel \psi$. Hence $\Pi \models \langle r_i\rangle(\phi \parallel \psi)$.

Theorem 5 (Soundness of axiom A6). $\models K_A\phi \wedge K_A(\phi \to \psi) \to K_A\psi$

Proof. Suppose that $\Pi \models K_A\phi$ and that $\Pi \models K_A(\phi \to \psi)$, where $int(A) = \Pi_1$. Then $\Pi = \Pi_1|\Pi_2$ and for any Π' we have $\Pi_1|\Pi' \models \phi$ and $\Pi_1|\Pi' \models \phi \to \psi$. Hence for any such $\Pi_1|\Pi'$ we have $\Pi_1|\Pi' \models \psi$ and because $\Pi = \Pi_1|\Pi_2$ we obtain that $\Pi \models K_A\psi$.

Further we present some meaningful theorems that can be derived with our system.

Theorem 6. $\vdash K_A\phi \to K_A\top$.

This theorem says that an observer knows something only if it is active.

Theorem 7 (Monotonicity of knowledge). *If* $\vdash \phi \to \psi$ *then* $\vdash K_A\phi \to K_A\psi$

The knowledge is monotone, meaning that if a property ϕ guarantees a property ψ then any observer that knows ϕ knows also ψ.

Theorem 8 (Consistency of knowledge). $\vdash K_A\phi \to \neg K_A\neg\phi$.

This theorem states that the knowledge of an observer is always consistent; the observer cannot know ϕ and $\neg\phi$.

Theorem 9 (Ontological dependency). *If* $int(A) = \Pi_1|\Pi_2$, $int(A_1) = \Pi_1$ *then* $\vdash K_A\top \to K_{A_1}\top$.

If the system associated to observer A_1 is a subsystem of the system associated to observer A, then the activation of observer A implies the activation of observer A_1.

For more interesting theorems, the reader is referred to [15, 16, 14], where, for this logic, it is also developed a semantics on process algebras proved to be sound and complete against the same axiomatic system.

6 A (Simple) Case Study

Consider the membrane system defined as:

Π :	Π' :	Π'' :
$\mu = [\,[\,]_2\,[\,]_3\,[\,]_4\,]_1$	$\mu' = [\,[\,]_2\,[\,]_3\,]_1$	$\mu'' = [\,[\,]_4\,]_1$
$w_1 = \lambda$	$w_1 = \lambda$	$w_1 = \lambda$
$w_2 = a$	$w_2 = a$	$w_4 = c$
$w_3 = b$	$w_3 = b$	
$w_4 = c$		
$R_1 = \{r' : b \longrightarrow b_{in4}\}$	$R_1 = \{r' : b \longrightarrow b_{in4}\}$	$R_1 = \{r' : b \longrightarrow b_{in4}\}$
$R_2 = \{r'' : a \longrightarrow b_{out}\}$	$R_2 = \{r'' : a \longrightarrow b_{out}\}$	$R_4 = \{r^{IV} : b \longrightarrow c_{out}\}$
$R_3 = \{r''' : b \longrightarrow a_{out}\}$	$R_3 = \{r''' : b \longrightarrow a_{out}\}$	
$R_4 = \{r^{IV} : b \longrightarrow c_{out}\}$		

Obviously $\Pi = \Pi'|\Pi''$. Suppose now that we have an observer $A \in Obs$ that can see only the membrane system Π', i.e., $int(A) = \Pi'$. Hence, for such observer, the system Π is an open one, as A can see the subsystem Π' and, for the rest, A accepts any other system as a possible one.

Suppose now that, using the knowledge of A, we want to compute the truth value of the following *property*: if Π contains a membrane 4 then, eventually, it is possible to send an object b to the membrane 4 (more exactly after two transitions). We can express this by stating (and proving) that the next formula can be derived, as axiom, from the presented axiomatic system.

$$\vdash K_A\top \rightarrow K_A([\![[\![\top]\!]_4 \parallel \top]\!]_1 \parallel \top \rightarrow \langle r''_2\rangle\langle r'_1\rangle([\![[\![b]\!]_4 \parallel \top]\!]_1 \parallel \top))$$

Indeed, the main precondition $K_A\top$ ensures that the observer A can see something in the system Π (i.e., Π' is a subsystem of Π). This implies that A knows

$$[\![[\![\top]\!]_4 \parallel \top]\!]_1 \parallel \top \rightarrow \langle r''_2\rangle\langle r'_1\rangle([\![[\![b]\!]_4 \parallel \top]\!]_1 \parallel \top)$$

We can read the knowledge of A as: if $[\![[\![\top]\!]_4 \parallel \top]\!]_1 \parallel \top$, meaning if the membrane 1 contains a membrane 4 and maybe something else then

$$\langle r''_2\rangle\langle r'_1\rangle([\![[\![b]\!]_4 \parallel \top]\!]_1 \parallel \top).$$

The fact that we are not interested in what membrane 4 contains it is expressed by the firsts two $\top$, while the fact that membrane 1 might also contains other things it is specified by the second $\top$.

Now, this post condition can be read as: the system can use the rule r'' in region 2 (which sends an object b to region 1), then it can apply the rule r' in region 1 (because now, in region 1 there is one b) and after doing these two transitions, we obtain a membrane system having membrane 4 inside membrane 1 and region 4 contains the object b. The two $\top$ are used for specifying the fact that in region 4, as well as in region 1, might be also other things in which (in this case) we are not interested in.

Following these steps the specified property can also be proved inside the syntax of the presented logic.

The important point is that we have succeeded to play with partial information *without using a complete description of the system Π, but only using the "point of view" about the system of the observer A*. Moreover, the specified property is true not only for the system Π, but also for any other system which looks to A "indistinguishable" from Π, i.e., any system of type $\Pi'|\Pi'''$ where Π''' is an arbitrary membrane system.

Indeed, if Π''' does not contain the membrane 4 then

$$K_A([\![[\![\top]\!]_4 \parallel \top]\!]_1 \parallel \top \rightarrow \langle r_2''\rangle\langle r_1'\rangle([\![[\![b]\!]_4 \parallel \top]\!]_1 \parallel \top))$$

is still true, as $[\![[\![\top]\!]_4 \parallel \top]\!]_1 \parallel \top$ is false, and in classic propositional logic false implies anything. From the other side if Π''' contains a membrane 4, then does not matter what else it contains, the system $\Pi'|\Pi'''$ it is still able to send a b in membrane 4 in two transitions, as just shown.

7 Conclusion

The logic we have proposed allows us to specify and formally prove properties of open membrane systems or, in general, properties that involve partial knowledge. Such properties cannot be formally described (in an "easy" way) by using the classic theory of membrane systems. The main idea of the presented logic is that it allows the analysis of the partial knowledge by collecting the partial information collected by observers of a membrane system. As showed in the example presented in Section 4, the logic allows to compute information by using logical reasoning on the information collected by the observers (even if they do not communicate each other). Sometime, using the presented logical tools, it is possible to interpret the "behavior" of the single observers for understanding the information we are looking for. Since this is done is a "distributed" fashion, this type of analysis has a computational price much smaller then the one needed for an analysis of the entire system.

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